FUNCTION AND ARGUMENT IN *BEGRIFFSSCHRIFT*

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ABSTRACT. It is well known that the formal system developed by Frege in *Begriffsschrift* is based upon the distinction between function and argument – as opposed to the traditional distinction between subject and predicate. Almost all of the modern commentaries on Frege's work suggest a semantic interpretation of this distinction, and identify it with the ontological structure of function and object, upon which *Grundgesetze* is based. Those commentaries agree that the system proposed by Frege in *Begriffsschrift* has some gaps, but it is taken as an essentially correct formal system for second-order logic: the first one in the history of logic.

However, there is strong textual evidence that such an interpretation should be rejected. This evidence shows that the nature of the distinction between function and argument is stated by Frege in a significantly different way: it applies only to expressions and not to entities. The formal system based on this distinction is tremendously flexible and is suitable for making explicit the logical structure of contents as well as of deductive chains.

We put forward a new reconstruction of the function-argument scheme and the quantification theory in *Begriffsschrift*. After that, we discuss the usual semantic interpretation of *Begriffsschrift* and show its inconsistencies with a rigorous reading of the text.

1. Introduction

Gottlob Frege's foundational work, *Begriffsschrift, eine der arithmetischen nachgebildete Forlmelsprache des reinen Denkens* [Frege, 1879], has been studied intensely over the last decades[1]. Throughout this period a unitary interpretation has been established and generally shared by scholars – with the exception of certain particular cases. One specific element of this interpretation is a particular understanding of the distinction – developed in *Begriffsschrift* – between function and argument. A careful reading of the

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[1]From now on and throughout the whole paper, for the sake of clarity, we will use '*Begriffsschrift*' to refer to the book published by Frege in 1879 and 'concept-script' to refer to the formal system developed in it.

sources shows not only that this usual understanding of the distinction has some inaccuracies, but that it is fundamentally incorrect.

When Frege presents the notion of function in *Grundgesetze der Arithmetik* [Frege, 1893] – from now on, *Grundgesetze* –, he acknowledges that his understanding of this notion has been modified from 1879 to 1893:

> "My *Begriffsschrift* (Halle, a.S. 1879) no longer corresponds entirely to my present standpoint; it is therefore to be consulted as an elucidation of what is presented here [*Grundgesetze*] only with caution." [Frege, 1893, §11, p. 5, footnote 1]

We will argue that the modifications this new understanding involve imply an essentially different notion of function from that of 1879. This is precisely what has been systematically minimised – if noted at all – in contemporary readings of *Begriffsschrift*. Our aim is to faithfully depict Frege's account of the distinction between function and argument that he developed in 1879.

We will propose a new reconstruction of the function-argument scheme and the quantification theory developed in *Begriffsschrift*. After the present introduction, in the second section, we will, on the one hand, state how an analysis in terms of function and argument is applied, and what characterises it; and, on the other, lay out the notion of generality. We then leave the expository part of the paper and begin the historical discussion. We will clarify, in the third section, the linguistic nature of the function-argument distinction and the relation between this structure and the elements of the language of the concept-script. In the fourth section, we will discuss the nature of the notion of generality and raise specific considerations concerning the syntactical guidelines Frege himself provides to handle it. Finally, in the fifth section, we will show that the reading of *Begriffsschrift*'s quantification done by most modern scholars is untenable.

2. Function-argument and generality: Exposition

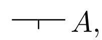
2.1. **Basic notions of the concept-script.**

Before considering the distinction between function and argument, we lay down some terminology. Frege divides the symbols of the concept-script into those that have a fixed meaning and those that express generality over different objects [Frege, 1879, §1, p. 111]. The former group includes the logical symbols; the latter consists of what Frege calls 'letters' and is the group that we will consider first.

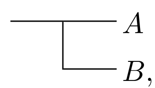
Letters express generality and have no determinate meaning. For explanatory reasons we differentiate between two sorts of letters: 'f','g','h',... are *function letters*; and 'a','b','c',... – which do not receive a specific name from Frege – will be referred to as *argument letters*. Our terminology is, in any case, in agreement with Frege's use.

The logical symbols of the concept-script are the content and the judgement strokes, the conditional and the negation strokes, the equality symbol and the generality symbol. The content stroke —— indicates that the content of certain combination of symbols is taken as a whole, and the judgement stroke | expresses the act of assertion by which a content is affirmed. Frege does not accurately explain what is the content. For our purposes, it will suffice to say that the content of a statement is what it means – an assertible content –, and that the content of a term is its denotation.

The conditional and the negation concern contents. If A and B are assertible contents, then the negation of A is denoted by:



and the conditional "if B, then A" is rendered by:



which links an antecedent B and a consequent A.

The equality symbol differs from the conditional stroke and the negation stroke in that it relates names, and not contents [Frege, 1879, §8, p. 124]. For instance, '$2 \cdot 3 \equiv 5 + 1$' expresses that '$2 \cdot 3$' and '$5 + 1$' have the same content, i.e. denote the same object[2]. We will later discuss in detail the generality symbol and its meaning in the concept-script.

2.2. **Function-argument analysis in *Begriffsschrift*.** In *Begriffsschrift*, the distinction between function and argument is explained with the help of simple expressions. From a technical point of view, a simple expression has neither quantification nor propositional structure: it is an atomic statement or an individual term. In his examples, Frege borrows expressions from natural language. We will also use concept-script expressions and arithmetical examples.

Frege introduces the function-argument scheme as a particular decomposition of expressions. According to his exposition, an expression can be decomposed into two components: a variable part, which is the component that can be replaced (the *argument*) and a fixed part (the *function*) [Frege, 1879, §9, p. 126].

The main characteristic of this distinction is the fact that it does not obey any pre-established guideline: any component can be the argument. It is taken for granted that the division has some reasonable restrictions, which are mainly dictated by common sense[3]. A prominent feature of this distinction is the absence of a rule for drawing the division. In Frege's words:

[2]As a consequence of Frege's rendering of the equality of content, this relation introduces an inconsistency in the interpretation of the letters of the concept-script. According to his exposition, the same symbol 'a' would be interpreted as a content – outside an equality statement – and, at the same time, – in an equality statement such as '$a \equiv b$' – as a symbol. We will not consider the difficulties this inconsistency could produce. For a detailed discussion of this matter, see 'Frege's *Begriffsschrift* Theory of Identity' [Mendelsohn, 1982] and 'What Frege's Theory of Identity is Not' [May, 2012].

[3]Each component must be a significative unit; for instance, articles or prepositions cannot be taken as a function or as an argument.

> "This distinction has nothing to do with the conceptual content, but only with our way of viewing it." [Frege, 1879, §9, p. 126]

This means, on the one hand, that the division between function and argument does not reflect the semantical structure of the expression where it is applied and, on the other, that there may be different ways in which an expression can be decomposed; each decomposition depends only on our particular interests.

For instance, the statement '3 is an odd number' can be decomposed in at least two different ways:

(1) The function 'is an odd number' and the argument '3'.
(2) The function '3' and the argument 'is an odd number'.

Frege provides a somewhat rigorous definition of the distinction between function and argument:

> "*If, in an expression (whose content need not be assertible), a simple or a complex symbol occurs in one or more places and we imagine it as replaceable by another (but the same one each time) at all or some of these places, then we call the part of the expression that shows itself invariant a function and the replaceable part its argument*[4]." [Frege, 1879, §9, p. 127]

The function-argument scheme can thus even be applied to non-assertible expressions. Therefore, any complex term – as well as any simple formula – can be divided into function and argument.

The letters of the concept-script are not suitable for informally explaining the meaning of the logical symbols of this formal system, because they express generality. This is the reason why, in addition to them, Frege uses capital Greek letters in order to represent particular contents without further determination: 'A','B','Γ','Δ',... and 'Φ','X','Ψ',.... These symbols do not belong to the language of the concept-script and neither do they express generality. Frege uses them mainly in chapter I of *Begriffsschrift* to assist his explanations of the components of the formal system – as we have done, for instance, for the conditional.

According to the use of these symbols, '$\Phi(A)$' represents a particular expression that has already been divided into two parts: under an initial analysis, 'Φ' is the function of the argument 'A'. Moreover, Frege provides a reading for '$\Phi(A)$', namely, 'A has the property Φ' [Frege, 1879, §10, p. 129]. This reading comes from the basic sentence represented by '$\Phi(A)$', which expresses that an object has certain property – either simple or complex. Hence, A takes the place of the object and Φ represents the property. However, since '$\Phi(A)$' is a generic scheme for any expression decomposed into function and argument, this reading becomes merely suggestive. Therefore, from a purely logical point of view, it does not impose any pre-established way to

[4]We have removed the expressions enclosed in brackets Terrell W. Bynum adds in his translation. We will do the same in what follows. All remaining brackets have been added by the authors.

handle the decomposition. In fact, as Frege suggests, the initial analysis of '$\Phi(A)$' – according to which 'Φ' is the function of the argument 'A' – can be revised:

> "Since the symbol Φ occurs at a place in the expression
>
> $$\Phi(A)$$
>
> and since we can think of it as replaced by other symbols Ψ, X – through which other functions of the argument A are expressed[5] – *we can consider $\Phi(A)$ as a function of the argument Φ*."
> [Frege, 1879, §10, p. 129, emphasis added]

In other words, the analysis in terms of function and argument is not absolute, and can be changed afterwards: once a division has been made, what was taken to be the replaceable part might become the fixed component and both symbols can thus be taken as either function or argument. Accordingly, nothing prevents us from considering 'Φ' as the argument of '$\Phi(A)$'. Consequently, in the expression '$\Phi(A)$' we can regard, on the one hand, 'Φ' as the function having 'A', 'B', 'Γ',... as arguments and, on the other, 'A' as the function having 'Φ', 'X', 'Ψ',... as arguments.

As we have said, expressions such as '$\Phi(A)$' are useful for the sake of explanation, but they do not belong to the language of the concept-script. In order to express properly in the language a generic expression divided into function and argument, the concept-script provides '$f(a)$'.

Frege adds an important modification to this scheme: the possibility of a function having more than one argument [Frege, 1879, §9, p. 128]. This provides – for the first time in the history of logic – an appropriate way to analyse relational statements.

The considerable flexibility with which the function-argument structure can be applied to simple cases is not suitable for all expressions. Frege warns of the inadequacy of applying the distinction between function and argument when taking the gramatical structure as a guide. This inadequacy is especially manifest in the decomposition of quantified statements. In order to demonstrate this, Frege uses the following examples [Frege, 1879, §9, p. 127]:

(3) The number 20 can be represented as the sum of four squares.
(4) Every positive integer can be represented as the sum of four squares.

These two statements exhibit a clear difference: they have a radically distinct subject, in spite of sharing the same predicate. According to Frege, there is a relevant element underlying this linguistic divergence:

> "What is asserted of the number 20 cannot be asserted in the same sense of "every positive integer"; though of course, in some circumstances it may be asserted of every positive integer. The expression "every positive integer" by itself, unlike "the number

[5] We do not follow here Bynum's translation. He renders this phrase "(...) replaced by other symbols [such as] Ψ, X – which then express other functions of the argument A".

> 20", yields no independent idea; it acquires a sense only in the context of a sentence." [Frege, 1879, §9, p. 128]

Frege seems to recognise the difference between these two statements in the fact that 'the number 20' denotes a particular object, while 'every positive integer' does not have a specific denotation, but refers generally to a variety of entities, namely, the positive integers.

The difference between these two statements is generalised with the help of the distinction between being determinate and being indeterminate:

> "For us, the different ways in which the same conceptual content can be considered as a function of this or that argument have no importance so long as function and argument are completely determinate. But if the argument becomes *indeterminate* (...), then the distinction between function and argument acquires a *substantive* significance. It can also happen that, conversely, the argument is determinate, but the function is indeterminate. In both cases, through the opposition of the *determinate* and the *indeterminate* or the *more* and the *less determinate*, the whole splits up into *function* and *argument* according to its own content, and not just according to our way of looking at it." [Frege, 1879, §9, p. 128]

Even though Frege does not specify what it means to be determinate, it can be said that an indeterminate component is either an argument or a function that expresses generality and has no fixed denotation. The meaning of indeterminate expressions cannot be expressed in the concept-script without letters. In contemporary terms, this is equivalent to the presence of either variables – in terms or in open formulas – or quantified fragments of an expression, such as 'every positive integer', which do not express an independent idea[6].

Every quantified statement contains indeterminate components. As Frege states, the distinction between function and argument has to be drawn according to the content in those expressions that contain indeterminate components. The reason for this variation is the need to avoid incorrect analyses, which are suggested by the gramatical structure of many expressions.

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[6]We can find at least three different uses of the pair '*bestimmt-unbestimmt*' in *Begriffsschrift*, which is commonly translated by 'determinate-indeterminate'. The distinction between the uses relies on their application. We have already explained the first use.

According to the second use, in the case of an expression such as '$\Phi(A)$', the symbol 'Φ' is indeterminate because it represents a particular expression – which is, under an initial analysis, taken as the function of '$\Phi(A)$' –, but without specifying which one.

Finally, a property is indeterminate if it is denoted by an expression such that, when the latter is taken as the function of an statement, it is not the case that the result of combining this function with any argument denotes an assertible content. There are cases of properties for which the expression of the circumstance that an object has them is not always a judgement. Frege's own example is the property of being a heap of beans [Frege, 1879, §27, p. 177].

Hence, Frege deals with the decomposition of quantified statements in a specific way in order to impose a correct logical analysis of the content that the concept-script manipulates. Now, it is not Frege's aim in *Begriffsschrift* to show how logically analyse content – although it certainly helps to do so. Rather, the aim of *Begriffsschrift* is to offer, once the content has been appropriately analysed, an adequate way of expressing it; a way which in turn is free of the ambiguities of natural language. Thus, at this stage of explanation, Frege does not specify how this particular analysis must be performed: he only warns that the general decomposition is not adequate for those expressions that contain indeterminate components.

After this exposition, we can state that 'the number 20' is a determinate expression and 'every positive integer' an indeterminate one. We know that 'the number 20' can be the argument of the function 'can be represented as the sum of four squares'. Hence, in (3), the most natural argument coincides with the subject. In this sense, the subject-predicate analysis is innocuous with respect to the general decomposition, since the latter can be applied without guidelines – just depending on the point of view adopted. However, as we have already pointed out, it is not appropriate to replicate this same decomposition with (4), for a previous formal analysis has to be carried out. This reveals the fact that, since the predicate 'can be represented as the sum of four squares' is taken to be simple – although actually it should be analysed –, (3) is an atomic expression while (4) is a complex quantified statement. Therefore, (4) must be analysed taking its content into account. This leads to a symbolisation such as that of universal affirmative statements made nowadays.

2.3. **Generality.** The generality symbol of the concept-script is related to a letter and, in Frege's words, "*delimits the scope of the generality signified by this letter*" [Frege, 1879, §11, p. 131]. This symbol is composed of two different elements. First, a concavity is placed in the middle of a content stroke – not necessarily attached to a judgement stroke – in such a way that its position indicates the scope of the the generality expressed. Second, the German font points out which letter is bound by the concavity. When a concavity occurs, '$\mathfrak{F}$' will be used as function letter and '$\mathfrak{a}$','$\mathfrak{b}$,'$\mathfrak{d}$','$\mathfrak{e}$' as argument letters. We will conventionally refer to the generality symbol as 'quantifier' and thus mention 'quantified statements'[7].

It is clear from the very presentation of this notion in *Begriffsschrift* how it is deeply linked with the function-argument scheme. According to Frege's proposal, a symbol in the expression of a judgement – either a function letter or an argument letter – is taken to be the argument and is replaced with a German letter [Frege, 1879, §11, p. 130]. In fact, he explicitly says that the function is what remains when a symbol in an assertible expression is

[7]This convention, however, should not induce to conclude that statements that do not contain any concavity are not quantified in our contemporary sense. Our account of Frege's use of italic letters makes this explicit.

regarded as the argument and a German letter replaces it. Thus, there is no particular link between the predicate of a quantified statement and the component taken to be the function.

We have said that the following statement:

$$\vdash\!\!\!-\!\!-\, \Phi(A)$$

can be analysed in a natural way taking 'A' to be the argument and 'Φ' to be the function. Hence, if the symbol 'A' is seen as the variable part, then it can be replaced with a corresponding German letter and thus be quantified. The result:

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, \Phi(\mathfrak{a}),$$

stands for the judgement that $\Phi(A)$ is a fact – that is, $\Phi(A)$ is true – whatever argument 'A' may be put in place of '$\mathfrak{a}$'. Frege also mentions that 'Φ' can be the argument as well, and it too can be thus replaced with the German '$\mathfrak{F}$' in order to get:

$$\vdash\!\!\overset{\mathfrak{F}}{\smile}\!\!-\, \mathfrak{F}(A).$$

We will consider this matter in detail in section 4.1.

The concavity and German letters are not the only means to render generality in the concept-script; italic letters express generality as well, but their scope encompasses the totality of the judgement where they occur. In Frege's words, "*An italic letter is always to have as its scope the content of the whole judgement*, and this need not be signified by a concavity in the content stroke" [Frege, 1879, §11, pp. 131–132]. In fact:

$$\vdash\!\!-\!\!\!\begin{array}{l} \top\!\!-\!\!-\, a > b \\ \llcorner\!\!-\!\!-\, a + b > b + b \end{array}$$

stands for the same judgement as:

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\!\!\overset{\mathfrak{b}}{\smile}\!\!-\!\!\!\begin{array}{l} \top\!\!-\!\!-\, \mathfrak{a} > \mathfrak{b} \\ \llcorner\!\!-\!\!-\, \mathfrak{a} + \mathfrak{b} > \mathfrak{b} + \mathfrak{b}. \end{array}$$

3. The nature of the distinction function-argument

We have so far considered Frege's exposition in *Begriffsschrift* of the distinction between function and argument, as well as his notion of generality. We will now depart from this expository approach in order to contrast our results with the relevant aspects of the traditional reading of *Begriffsschrift*[8].

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[8]Throughout this section, for the sake of simplicity we consider only unary functions in order to set our position.

3.1. **Functions in *Begriffsschrift* and mathematical functions.** Probably because of the groundbreaking character of Frege's 1879 work, many modern historical commentaries tend to identify the notion of function in *Begriffsschrift* with a modern one – which has strong similarities with the notion of mathematical function: an assignment of a unique value to every appropriate argument[9]. The notion of function in *Begriffsschrift* is relative in the sense that a component can be either a function or an argument of a single expression, whereas the mathematical notion of function is absolute, that is, function and argument are not interchangeable in the same sense. The divergence between the way in which Frege introduces functions in *Begriffsschrift* and modern accounts is often considered as mere inaccuracy.

Frege indicates that the notion of function developed in *Begriffsschrift* takes as a guide the notion of mathematical function, but at the same time he points out that there are great differences between the two notions:

> "[S]ince we can think of it [Φ in $\Phi(A)$] as replaced by other symbols Ψ, X – through which other functions of the argument A are expressed[10] – we can consider $\Phi(A)$ as a function of the argument Φ. This shows quite clearly that the concept of function in [mathematical] analysis, which I have in general followed, is far more restricted than the one developed here." [Frege, 1879, §10, p. 129]

While the reading of an expression such as '$\Phi(A)$' – divided into function and argument – is, in Frege's words, 'A has the property Φ', he would never read the mathematical expression '$f(x)$' as 'x has the property f'[11]. And, contrary to what happens with $\Phi(A)$, f could never be the mathematical argument of $f(x)$. In this sense, the *Begriffsschrift* function cannot be seen as a generalisation of the function in mathematical analysis. In other words, the divergence between these two notions is not a matter of an extension of the domain of possible arguments and values.

What Frege takes from analysis is the fact that in every arithmetical expression of a function – such as, for instance, '$2x^2 + 3$' – there is always a component that takes values, 'x', while the remaining component is fixed. Accordingly, function and argument in *Begriffsschrift* are only loosely related to their arithmetical counterparts[12].

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[9]Gordon Baker and Peter Hacker firmly and repeatedly defend this possibility in 'Functions in *Begriffsschrift*' [Baker; Hacker, 2003]. More recently, in *Gottlob Frege: A Guide for the Perplexed* [Kanterian, 2012, pp. 128–129], Edward Kanterian contemplates the need for a *Begriffsschrift* function to yield a value – either an object in the universe or a truth-value – once combined with a suitable argument.

[10]See footnote 5.

[11]This is exactly the aspect Frege emphasises when he compares these two concepts of function in 'Booles rechnende Logik und die Begriffsschrift' [Frege, 1880, p. 26].

[12]Our reconstruction of the two notions of function essentially coincides with the account by Carl Theodor Michaëlis in his review of *Begriffsschrift* [Michaëlis, 1880]:

> "The form of the function symbols is the same as the usual one of mathematics. It differs in sense from the mathematical one since it

The sense in which the function-argument scheme is more general than that of mathematical analysis is exemplified by Frege in 'Anwendungen der Begriffsschrift':

> "According to the more general conception of function [*allgemeinerer Funktionsbegriff*] that I took as a basis, we can regard
>
> $$u + 1 = v$$
>
> as a function of u and v and can therefore view it as a particular case of $f(u, v)$." [Frege, 1879b, pp. 204–205]

In analysis, '$u + 1 = v$' would never be seen as a function of two variables. However, it is perfectly natural for Frege to take this expression – according to *Begriffsschrift* – as a function of 'u' and 'v', that is, as '$f(u, v)$'.

There is another difference we want to consider regarding the two notions alluded here. In mathematical analysis, every function yields a unique number for every number. As a mathematician, Frege should know this and yet, for some reason, he decides not to incorporate this fundamental characteristic into his account of function in *Begriffsschrift*: in this work, a function is not an assignment of a value to an argument. Only later does Frege generalise the mathematical notion of function in such a way that it assigns objects – and not merely numbers – to objects, as he explicitly affirms in *Function und Begriff* [Frege, 1891, pp. 144–146][13].

3.2. **Function-argument vs. concept-object.** One of the first things we have highlighted about the distinction between function and argument is the fact that it is applied only to expressions – once they have been presented in an adequate way. We defend the claim that there is absolutely no hint in Frege's exposition which suggests that the field of applicability of this distinction is more than linguistic in nature[14]. According to our

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> signifies, not the whole of the dependent expression of magnitude, but, unlike the argument, only the invariant part of the expression. Also, the logical function symbol allows interchange of argument and function." [Michaëlis, 1880, p. 215]

This is specially relevant as Michaëlis was a contemporary of Frege.

[13]On some occasions, Frege uses the mathematical practice of taking $f(x)$ as a function *of* x in order to exemplify his function-argument distinction in *Begriffsschrift*; for instance, by saying that "we can consider $\Phi(A)$ as a function *of* the argument Φ" [Frege, 1879, §10, p. 129, emphasis added]. However, these examples are not proof that the *Begriffsschrift* functions yield values. First, they can be read coherently, according to our reconstruction, with the other of cases in which Frege expresses himself in different terms concerning the same distinction. Second, if there is a sense in which the combination of function and argument provides a value in *Begriffsschrift*, it is just the trivial result of taking the resulting expression of this combination as a value. It would be extremely difficult to clarify the rest of Frege's exposition of the function-argument dichotomy assuming the requirement for every *Begriffsschrift*'s function to yield a non-trivial value for each argument.

[14]Nevertheless, there has notoriously been controversy regarding the linguistic nature of the function-argument distinction in *Begriffsschrift.* Michael Dummett, on the one hand, and Baker and Hacker, on the other, are prominent contenders of the resulting polemic. See *Frege: Logical Excavations* [Baker; Hacker, 1984, pp. 104–144] and the succession of

position, there is no denotation in *pure* concept-script and hence there is no supra-linguistic dichotomy with which the function-argument scheme can be related[15]. Furthermore, the analogy between the function-argument dichotomy – developed in *Begriffsschrift* – and the distinction between concept and object – deployed in Frege's later works – is problematic. Although the two structures are clearly related, there is absolutely no reason to identify them[16].

In the previous section we have provided evidence against a particular absolute sense commonly associated with *Begriffsschrift*'s functions. We will now complement our discussion there by claiming that there is no absolute sense that can be attributed to the notion of function in *Begriffsschrift* according to which the function of an expression can be related with a concept or a property.

First of all, there is no direct correspondence in *Begriffsschrift* between a decomposition in terms of function and argument and the ontological structure expressed by a statement. To put it in another way, the former is not determined by the denotation of the components of an analysed expression.

We have already argued that the analysis in terms of function and argument is not absolute: what has been initially seen to be the function in a given expression can be taken to be the argument afterwards. This could never be the case if functions had an ontological counterpart. A direct consequence of this is that the function-argument scheme cannot reflect the semantic structure of atomic judgements. The expression 'Socrates is mortal' can be analysed in such a way that either 'Socrates' or 'is mortal' be the argument. In contrast, neither Socrates can be the concept of the atomic content

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reviews and replies its publication generated: 'An Unsuccessful Dig' [Dummett, 1984], 'Dummett's Dig: Looking-Glass Archeology' [Baker; Hacker, 1987], 'Reply to 'Dummett's Dig', by Baker and Hacker' [Dummett, 1988], and 'The Last Ditch' [Baker; Hacker, 1989]. Dummett argues that the distinction between function and argument in *Begriffsschrift* is not tied to any supra-linguistic structure. The opposite thesis, according to which the distinction between function and argument should be applied to the meaning of expressions, has been more recently defended by Baker in "Function' in *Begriffsschrift*: Dissolving the Problem' [Baker, 2001], by Baker and Hacker in 'Functions in *Begriffsschrift*' [Baker; Hacker, 2003] and by Michael Beaney in 'Frege's use of function-argument analysis and his introduction of truth-values as objects' [Beaney, 2007].

[15]We refer with 'pure concept-script' to the isolated use of this formal system, such as that of chapter II of *Begriffsschrift*. In pure concept-script, the only non-logical symbols of the language are letters. See section 3.3 for a treatment of this notion and a discussion about the differences between pure and applied concept-script.

[16]A careful historical analysis of the sources shows that the general approach of defending an analogy between Frege's mature position concerning the notions of function and object and his account of function and argument in *Begriffsschrift* – an analogy defended, for instance, for the sake of an expected uniformity in Frege's thought – is out of place. See the comments on this matter by Hans Sluga in *Gottlob Frege. The Arguments of the Philosophers* [Sluga, 1980, p. 139], Wolfgang Kienzler in *Begriff und Gegenstand* [Kienzler, 2009, p. 57] and, especially, Richard G. Heck and Robert May in 'The Function is Unsaturated' [Heck; May, 2013, pp. 826–827].

expressed by ‘Socrates is mortal’, nor can the property of being mortal be its object: that would not only be impossible, but also completely senseless. This shows that the analogy between a property or a concept and a function is in general impossible to sustain. In fact, when Frege tries to explain the distinction between concept and object or individual – as he does, just after the publication of *Begriffsschrift*, in ‘Booles rechnende Logik und die Begriffsschrift’ [Frege, 1880, pp. 16–18] –, he does not turn to the function-argument dichotomy[17].

On the second place, an absolute notion of function will not appear in a substantial way in Frege’s writings until 1891, in *Function und Begriff* [Frege, 1891]. The fact that Frege takes in this text the mathematical notion of function as a basis for a precise characterisation of the notion of concept has a singular relevance for our present discussion. It should not be minimised that until 1891 the author does not discover a means to rigorously associate functions and concepts. The completion of this association requires, in addition to taking truth-values to be objects, considering concepts and properties as functions – in the mathematical sense – from objects in the universe to objects in the universe. This is exactly what Frege claims to have accomplished in *Grundgesetze*:

> “Moreover, the nature of functions, in contrast to objects, is characterised more precisely [in *Grundgesetze*] than in *Begriffsschrift*. Further, from this the distinction between functions of first and second level results. As elaborated in my lecture *Function und Begriff*, concepts and relations are functions as I extend the reference of the term, and so we also must distinguish concepts of first and second level and relations of equal and unequal level.” [Frege, 1893, p. x]

If Frege had already pressed the analogy between the notion of function and that of property or concept in *Begriffsschrift* – as it is so often claimed –, then he would not speak of the extension of the reference of the term, but just of a matter of detail, and he certainly would not say that “*Begriffsschrift* (...) no longer corresponds entirely to my present standpoint” [Frege, 1893, §11, p. 5, footnote 1] concerning the notion of function in *Grundgesetze*. In the light of the previous text, this possibility is hard to accept[18].

[17]The fact that, in the papers elaborated straight after the publication of *Begriffsschrift*, Frege does not allude to the distinction between function and argument when he wants to render the concept-object structure is a further indication of the mistaken nature of Baker and Hacker’s claim in ‘Functions in *Begriffsschrift*’ that “he tied his concept of a function to concept formation” [Baker; Hacker, 2003, p. 283].

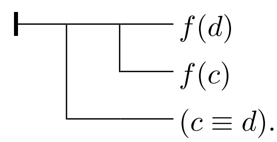
[18]Notice that the most prominent constituent of the Fregean notion of concept, namely, insaturation, is completely absent in *Begriffsschrift*. It appears for the first time in a letter to the philosopher Anton Marty (1847-1914) dated from 29th August 1882 [Frege, 1976, p. 101].

The differences between Frege’s perspective in *Begriffsschrift* and *Grundgesetze* – especially concerning the divergence between the function-argument scheme and the concept-object scheme – are seldom noticed and almost never properly considered.

Most of the arguments employed to defend the close analogy between the schemes function-argument and concept-object take the notion of conceptual content as essential. For instance, authors such as Baker, Kanterian, Mark Textor or Joan Weiner claim that there is a correspondence between the linguistic structure of expressions – articulated by the distinction between function and argument – and the semantic structure of their content[19]. Accordingly, Weiner argues that "the value of [the function] *is a planet* for the Earth as argument is, according to the views of *Begriffsschrift*, the conceptual content of the sentence 'The Earth is a planet'." [Weiner, 2004, p. 77].

However, the formulas of the concept-script cannot be linked to any particular semantic structure, because they acquire such a variety of readings that they cannot be assigned properly a definite meaning. We will see that the letters of the concept-script can have instances of very different kind.

Consider Proposition (52) of *Begriffsschrift* [Frege, 1879, §20, p. 161]:



(5)

This proposition is a basic law of the concept-script and has different readings. We will express them in a contemporary formal language.

First, 'c' and 'd' can be seen as individual variables; this is the usual interpretation when the concept-script is reconstructed as a system of first-order logic. In that case, the letter 'f' would be a formula variable and '$f(c)$' and '$f(d)$' formulas where 'c' and 'd' occur as free variables. This first reading is:

$$x = y \to (\phi(x) \to \phi(y)). \tag{6}$$

A particular case of this reading consists of taking 'f' as a predicate variable:

$$x = y \to (Xx \to Xy). \tag{6'}$$

Nevertheless, Heck and May identify in 'The Composition of Thoughts' [Heck; May, 2011, pp. 129–134] the reasons for the deep divergence in this matter that appears between *Begriffsschrift* and *Grundgesetze*. They also defend this view throughout 'The Function is Unsaturated' [Heck; May, 2013]. However, there is some tension in their analysis, as their fluctuating position regarding Frege's observance of the distinction between use and mention in *Begriffsschrift* shows. As opposed to Heck and May's account, our reconstruction of the notion of function in *Begriffsschrift* defends clearly that no absolute sense can be attributed to it and, at the same time, explains how this notion is related to the mathematical notion of function. Moreover, in the following sections we clarify the link between the function-argument scheme and the content expressed by an statement and suggest how this distinction constitutes *Begriffsschrift*'s notion of generality. All these elements are absent in Heck and May's account.

[19]See "Function' in *Begriffsschrift*: Dissolving the Problem' [Baker, 2001, pp. 537–538], *Frege: A Guide for the Perplexed* [Kanterian, 2012, p. 139], *Frege on Sense and Reference* [Textor, 2011, pp. 76–77] and *Frege Explained* [Weiner, 2004, pp. 76–77], respectively.

Second, 'c' and 'd' can also be propositionally interpreted. Hence, 'f' cannot be a second-order variable at all. In fact, it has no equivalent in the formal language: it should be seen as an expression in which a sub-formula is replaced with a logically equivalent formula. Accordingly, (5) can also be read as:

If ϕ and ψ have the same content, then if $\Phi(\phi)$ then $\Phi(\psi)$.

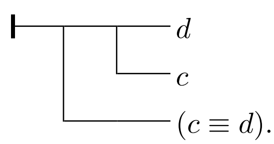
This formula could be specified as:

$$(\phi \leftrightarrow \psi) \rightarrow (\Phi(\phi) \rightarrow \Phi(\psi)). \tag{7}$$

Frege's equality symbol '$\equiv$' has two different readings in (6) and (7): either as an equality between terms or as a biconditional. We will see below an example taken from *Begriffsschrift* which shows that the reading of (5) as (7) agrees with Frege's practice.

The processes of substitution of the concept-script demonstrate the variety of kinds of instances a single letter can have. There is a uniformity in the substitutions performed in contemporary logic: individual variables are replaced with terms; propositional variables are replaced with complex formulas; and so on. However, Frege's practice in *Begriffsschrift* does not show the same uniformity; most substitutions involve – according to contemporary standards – a change in the interpretation of the original formula. Even though, from *Begriffsschrift*'s point of view, they just imply a restriction on the possible readings a single formula can have.

We will exemplify this last claim with a substitution performed by Frege in some derivations. The formula:



(8)

is obtained from the substitution of 'c' for '$f(c)$' and 'd' for '$f(d)$' in (5). These replacements can be schematised just by saying that Frege substitutes 'α' for '$f(\alpha)$' in (5)[20].

[20]In order to ease our explanation, we will use, from now on, small Greek letters – such as 'α', 'β', 'γ' – as place-holders for the argument in a complex function. This is especially useful in the explanation of the substitutions. Frege uses capital Greek letters – such as 'A', 'B', 'Γ' – for the same purpose. However, this introduces an ambiguity in the use of these symbols, since they are, on the one hand, taken as representatives of particular contents and, on the other, used to indicate how the argument and the function of a given expression must fit together. Frege does not distinguish between these two contexts.

The substitution of 'α' for '$f(\alpha)$' in (8) occurs in the derivation of Proposition (89) [Frege, 1879, §28, p. 181]; it is used by Frege only in those contexts where the argument letters are propositionally interpreted. According to this propositional interpretation, '$f(\alpha)$' is a formula in which 'α' occurs: in particular, it can be 'α'. A modern translation of (8) would be:

$$(\phi \leftrightarrow \psi) \rightarrow (\phi \rightarrow \psi).$$

While (5) can be read as (6) or (7), the formula (8) only allows a propositional interpretation. Now, through the transition from (5) to (8), we see that it is difficult to determine in general the kind of instances 'f' can have in (5): this letter is eliminated in the substitution that yields (8) – which is considered by Frege as an acceptable replacement – and hence can hardly be taken as a letter that stands for properties.

We can draw two conclusions from this last remark. First, a single letter of the concept-script must not have instances of a single and fixed kind. On the contrary, it can be interpreted in a wide variety of ways[21]. Among the letters of the concept-script there is no distinction between those expressing generality over predicate symbols – symbols that denote properties – and the rest. Moreover, there is no substantive and invariable role a letter will play in all particular analyses in terms of function and argument. A function letter can be the argument of a proposition and an argument letter can be taken to be the function in some contexts. This does not preclude the fact that an initial decomposition of an expression such as '$f(a)$' will take 'f' to be the function and 'a' to be the argument. For this reason we have used the denominations 'function' and 'argument' to refer to these two sorts of letters.

Second, the variety of possible readings that a single letter of the concept-script acquires definitely shows the enormous generality such a letter expresses, which Frege not only does not mean to avoid, but indeed is seen by him as a tremendous advantage. This generality is a direct proof of the plasticity the letters of the concept-script exhibit and also a strong argument against the thesis that the expressions of the concept-script properly have a definite meaning – whose structure is reflected in their linguistic decomposition.

3.3. **Application of the concept-script.** The usual discussion concerning the nature of functions in *Begriffsschrift* is almost exclusively based on Frege's explanation of this notion in chapter I of this book. However, this introductory chapter does not contain the concept-script proper. Frege uses examples of natural language for explanatory purposes, but those examples are often taken by advocates of a supra-linguistic interpretation of the function-argument

[21] This conclusion might be reinforced with the following formula:

$$\begin{array}{l} \vdash\!\!\!\!-\!\!\!\!-\!\!\!\!\top\!\!\!-\!\!\!-\, (f(c) \equiv f(d)) \\ \qquad\;\; \llcorner\!\!-\!\!-\, (c \equiv d). \end{array} \tag{9}$$

It does not appear in *Begriffsschrift*, but it can easily be shown to be a theorem of the concept-script. While readings that are analogous to (6) and (7) can be extracted from (9), there is another possible reading. Additionally, the letter 'f' can be interpreted as a function variable. In that case, 'c' and 'd' are taken as individual variables. Hence, the formula (9) would also be read as:

$$x = y \to fx = fy.$$

structure as the only representatives – and as the unique object – of Frege's analysis. If they were indeed the case, the semantical interpretation of the distinction between function and argument those particular examples suggest would be reproducible in pure concept-script; but it is not. Even if commentators such as Baker claim that Frege's exposition in §§9–10 of *Begriffsschrift* must be applied to judgements expressed in the language of the concept-script, their arguments do not actually apply to pure concept-script, but to a certain complementation of its language and that of a particular subject matter[22].

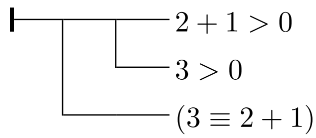
The language of the concept-script should not be seen as a formal language in the contemporary sense, but as a device for the accurate expression of the logical relations that link statements from a given subject matter – paradigmatically, arithmetic. From this perspective, this formal system – leaving aside provisionally what Frege does in chapter III of *Begriffsschrift* (see section 4.2) – can be seen as a tool for developing a discipline such as arithmetic, so that it provides neither a formalisation nor a reduction. The concept-script, on the one hand, allows to express all logical relations between arithmetical statements; and, on the other, makes manifest the fact that any mathematical proof can be formulated as a series of strictly regulated steps, regimented with a small set of inference rules. As a result, the concept-script eliminates all trace of natural language from arithmetic. The product of this complementation was known as *logistics*[23].

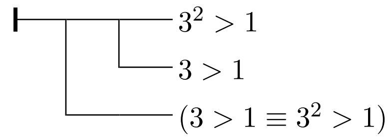
In logistics, the symbols of a particular discipline are mixed with the symbols of a formal language – and, in this case, with the concept-script symbols. The result is a regimented language to which the formal resources of the concept-script can naturally be applied. In particular, all expressions can be analysed in terms of function and argument. Now, since the meaning of the statements of this scientific discourse is structured – according to its semantics – in terms of concepts, properties and objects, it is thus not surprising to find certain analogy between a function-argument decomposition and the semantical structure of a statement in the given subject matter.

If we consider expressions of pure concept-script, it should be acknowledged that their only non-logical symbols are letters and therefore that they lack interpretation. Such expressions do not appear in formalised arithmetic, but they are components of a sort of superstructure or metalanguage. Recall our formula (5). Since it is not expressed in a logistic language, there is no direct link between its symbols and any precise semantic content; the formula is meant to be complemented with the language of a discipline. Only the symbols of this applied language – and, consequently, its formulas – can rigorously express the specific meaning with which the discipline is concerned. In this sense, the relation between the formulas of pure concept-script and a

[22]See "Function' in *Begriffsschrift*: Dissolving the Problem' [Baker, 2001, pp. 528–529].
[23]See, for instance, [Lewis, 1918, p. 3].

specific meaning is merely indirect. An example will help us to shed light on our position. Consider the following instances of (5):





As this example shows, the plasticity exhibited by the formulas of the concept-script emerges as a tremendous advantage when this formal system is used as part of logistics. Accordingly, the possibility to naturally and fruitfully applying the formal resources of the concept-script to a particular discipline is due to the fact that the expressions of pure concept-script do not have any particular meaning and can be read in different ways.

4. Remarks on Frege's exposition of generality

4.1. **Reconstruction of Frege's account of generality.** Most modern commentators defend the notion that the quantifiers of the concept-script are interpreted over a general domain. Now, Frege's explicit formulation of the notion of generality is the following:

> "In the expression of a judgment we can always regard the combination of symbols to the right of $\vdash\!\!-\!\!-$ as a function of one of the symbols occurring in it. *If we replace this argument by a German letter and introduce in the content stroke a concavity containing the same German letter, as in*
>
> $$\vdash\!\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a}),$$
>
> *then this stands for the judgment that the function is a fact whatever we may take as its argument.* Since a letter which is used as a function symbol, like Φ in $\Phi(A)$, can itself be considered as the argument of a function, it can be replaced by a German letter in the manner just specified." [Frege, 1879, §11, p. 130]

After this explanation of the meaning of the quantifiers of the concept-script, Frege specifies certain conditions for the instances of a quantified letter. These conditions help to understand how the generality the 'whatever' expresses – that we interpret syntactically, as we defend in section 3.2 above – should be handled:

> "The meaning of a German letter is subject only to the obvious restrictions that [1] the assertibility (§2) of a combination of symbols following the content stroke must remain intact, and [2] if the German letter appears as a function symbol, this circumstance must be taken into account." [Frege, 1879, §11, p. 130]

Departing from the traditional interpretation of this fundamental element of the concept-script, we claim that, following what Frege establishes in these two quotations, his account of the quantification in *Begriffsschrift* can be systematised as follows:

(∗) If '$f(a)$' is any assertible expression where the letter 'a' is taken to be the argument:

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, f(\mathfrak{a})$$

stands for the judgement that $f(A)$ is a fact whatever argument 'A' may be put in place of '$\mathfrak{a}$' preserving the assertibility of the expression.

This way of reconstructing quantification can be naturally applied to any kind of quantified expression. In particular, Frege introduces a new function letter – 'F' –, which is used only in chapter III of *Begriffsschrift*. As Frege mentions, function letters such as 'F' can be taken to be arguments as well and hence be replaced with a corresponding German letter[24]. In this sense, a quantification over function letters should be seen as a particular case of (∗). Accordingly:

(∗∗) If '$f(F)$' is any assertible expression where the letter 'F' is taken to be the argument:

$$\vdash\!\!\overset{\mathfrak{F}}{\smile}\!\!-\, f(\mathfrak{F})$$

stands for the judgement that $f(B)$ is a fact whatever argument 'B' may be put in place of '$\mathfrak{F}$' preserving the assertibility of the expression.

The transition from (∗) to (∗∗) is just a matter of application. Let us see an example of this kind of application. The following formula is applied by Frege:

$$\begin{array}{l} \vdash\!\!\!-\!\!\!-\!\!\!\!\!\!\begin{array}{l} \!\!-\!\!\!\!\begin{array}{l} \!\!-\!\!-\, f(c) \\ \llcorner\!-\, b \end{array} \\ \llcorner\!-\!-\, \big[(-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, f(\mathfrak{a})) \equiv b\big] \end{array} \end{array} \tag{10}$$

as follows:

$$\begin{array}{l} \vdash\!\!\!-\!\!\!-\!\!\!\!\!\!\begin{array}{l} \!\!-\!\!\!\!\begin{array}{l} \!\!-\!\!-\, f(F) \\ \llcorner\!-\, b \end{array} \\ \llcorner\!-\!-\, \big[(-\!\!\overset{\mathfrak{F}}{\smile}\!\!-\, f(\mathfrak{F})) \equiv b\big]. \end{array} \end{array} \tag{10'}$$

We have labelled Proposition (68) of *Begriffsschrift* as (10). It should be noted that, although Frege proves Proposition (68) as in (10), he then uses it – for instance, in the derivation of Proposition (77) [Frege, 1879, §27, p. 174] – as in (10′), which is a particular case of (10). Observe that the following expression:

$$-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, f(\mathfrak{a})$$

is any quantified formula and, accordingly, that '$\mathfrak{a}$' represents any argument. In particular, '$\mathfrak{a}$' can be a function symbol as well – as it is made explicit in

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[24]The first use of the quantification over function letters in *Begriffsschrift* can be found in Proposition (76) [Frege, 1879, §26, p. 173].

the transition from (10) to $(10')$[25]. We will discuss the relationship between the quantification of argument letters and that of function letters in section 4.2.

As we have said above, the distinction between function and argument is not absolute: a function symbol 'Φ' can be the argument of an expression such as '$\Phi(A)$', independently of the fact that – in other contexts – it would be considered its function. Now, since the quantification of 'Φ' in an expression such as '$\Phi(A)$' consists in a quantification of a function letter, it can be expressed in the language of the concept-script as follows:

$$\vdash\!\!\overset{\mathfrak{F}}{\smile}\!\!-\, \mathfrak{F}(a).$$

This stands for the judgement that $\Phi(a)$ is a fact whatever argument 'Φ' may be put in place of '$\mathfrak{F}$' preserving the assertibility of the expression[26].

Our reconstruction in $(*)$ can be used to explain Frege's restrictions to the meaning of a quantified letter. First, Frege introduces a general criterion for limiting the possible instances of a quantified letter: "the assertibility of a combination of symbols following the content stroke must remain intact" [Frege, 1879, §11, p. 130]. Accordingly, the acceptable instances are exactly those expressions that, once added to the functional component, yield an assertible expression. This criterion brings into question the possibility of there being unlimited quantification in the concept-script, say, over absolutely any meaningful expression.

The second restriction, as Frege puts it, establishes that "if the German letter appears as a function symbol, this circumstance must be taken into account" [Frege, 1879, §11, p. 130]. From a syntactical perspective, this case of quantification does not differ from quantification over argument letters: they both concern arguments. But there are two relevant circumstances that only affect function letters: their arity and certain specific conditions of assertibility applied to the judgements in which they occur.

The basic and minimal expression of the concept-script is a function letter with one – or several – argument letters enclosed in parentheses; for instance, '$f(a)$' or '$g(a,b)$'. Regardless of whether the function letter is the function or the argument of a given – simple or complex – formula, it always occurs with some symbols enclosed in parentheses. The number of these symbols determines the arity of the function letter. Frege points out that every instance of a quantified function letter must share the arity of this letter. Thus, for example, a binary function letter is not an acceptable instance of a unary letter. Moreover, as we will see in an example in section 4.2, in some contexts the function component of an expression has to adapt itself to the

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[25] We have indicated only some of the substitutions that affect (10). Specifically, Frege substitutes a complex instance for 'f'.

[26] An expression such as '$f(a)$' consists of two elements: either of them can be taken to be quantified, and thus be the argument. The case $(*)$ generically represents the quantification of an argument, whatever nature it has. Hence, this last form of quantification is only a particular case of $(*)$.

circumstance that a quantified function letter occurs with the number of symbols corresponding to its arity. This determines further delimitations for preserving the assertibility of the whole expression.

An example will help us to clarify our exposition concerning this last limitation. Some possible instances of the judgement:

$$\vdash\!\!\overset{\mathfrak{F}}{\smile}\!\!-\mathfrak{F}(a)$$

are the following:

$$\vdash\!\!\!-\!\!-\, f(a) \quad \text{or} \quad \vdash\!\!\!-\!\!-\, 2+a>3,$$

or even:

$$\begin{array}{l} \vdash\!\!\!-\!\!-\!\!\!\!\!\begin{array}{l}\top\!\!-\!\!- a \\ \,\,\,\lfloor\!\!-\!\!- a,\end{array}\end{array}$$

where the connective is an instance of the quantified letter '$\mathfrak{F}$'[27]. All these examples preserve the assertibility of the whole expression. Moreover, even the last instance can be schematised with $\vdash\!\!\!-\!\!-\,\Phi(A)$.

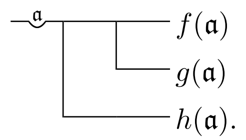
4.2. **Quantification and assertibility.** Our reconstruction of the notion of generality can be supplemented by focusing on two particular issues: the specificity of the quantification over function letters and the notion of assertibility.

First, there is no hierarchy of quantification in our reconstruction: $(**)$ is a particular case of $(*)$. The fact that we have made the case $(**)$ explicit is simply a convenient way of providing a specific account of function letters. This specificity should be clear in our discussion above. However, the particularities of the quantification over function letters by no means modify the unitary account Frege offers[28]. In fact, an example will show that the author has no substantive difference in mind between quantification over function letters and that over argument letters. In the derivations contained in chapters II and III of *Begriffsschrift*, Frege gives expressions which are to be substituted for letters in a given formula. Since letters express generality, we can view this process of substitution as an instantiation. Hence, in the process of derivation of a proposition and in the application of the substitutions, we find a clear and reliable testimony to the way Frege handles quantification. In the derivation of Proposition (93) [Frege, 1879, §28, pp. 182–183], Frege proposes certain modifications to one of its premisses – Proposition (60). For

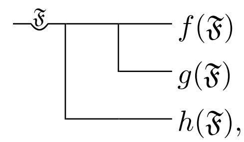
[27]But 'a' is not an acceptable instance of '$\mathfrak{F}$'.

[28]The fact that $(**)$ is a particular case of $(*)$ could have induced Susan Russinoff – who partly follows what George Boolos suggests in 'Reading the *Begriffsschrift*' [Boolos, 1985, p. 339] – to state in 'On the Brink of a Paradox?' [Russinoff, 1987, p. 128] that the quantification over function letters is a particular case of the quantification over argument letters. However, according to her account, function letters take values over concepts and relations, while argument letters take values over every entity in the universe. We will discuss such a pre-established domain for quantified letters in section 5.3 below.

the sake of simplicity, we will consider these changes only in the antecedent of (60):

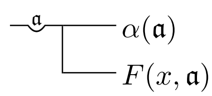


Frege applies this formula in the following way:

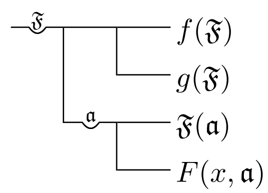


which results from replacing '𝔞' with '𝔉'.

The quantification of '𝔉' in this last formula has to be seen as an example of our general reconstruction: it consists of a specification of the argument. From a syntactical perspective, the reading of '𝔞' as '𝔉' involves the need to consider two circumstances: the arity of the function letter and the adaptation of the function to the new quantified argument. In the end, function letters have a specific place in the expressions of the concept-script. Accordingly, in the same derivation Frege makes certain substitutions in the function, that is, replaces some of the letters that occur in the fixed component with expressions that fit with the new argument. For instance, Frege substitutes:

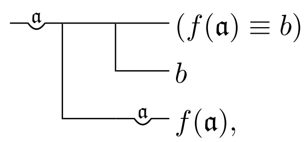


for '$h(\alpha)$'. Hence, the result of this change is:



In this substitution, we can appreciate both the preservation of the arity of '𝔉' in '𝔉(𝔞)' and the adaptation of the function – through an instance of 'h' – to the assertibility of the whole expression, i.e., to the circumstance that the argument is a function letter.

Second, we want to shed some light on the notion of assertibility. We mention this notion through the foregoing exposition without properly explaining how it is to be understood in *Begriffsschrift*. Frege does not discuss this matter further; even though it is a complex issue. In the concept-script there seem to only be syntactical rules for the evaluation of the assertibility of an expression; those rules establish, for instance, the correct formation of a complex formula or the acceptable instances of substitution of a letter. An example of a non-assertible expression in the concept-script can be:



for it produces a clash between the scope of two different quantifiers of the same German letter[29].

However, the notion of assertibility that is relevant in *Begriffsschrift* is more than syntactic. In chapter II of this work, the concept-script is handled in an abstract and isolated way. However, as we discuss in section 3.2, the concept-script is also meant to be used as the basis for logistics.

In light of this use, it is thus not surprising that there is no definition of atomic formula in *Begriffsschrift*: all atomic expressions – and the proper symbols necessary for expressing it – are provided by a certain subject matter. The language of the discipline this subject matter belongs to offers a set of symbols and syntactic rules for the formation of atomic expressions, which are supplemented by the formal resources of the concept-script. These make possible to form complex expressions or to render quantification rigorously. In consequence, the semantical and syntactical elements of the concept-script that remain to be clarified are resolved by the application of this formal system to some subject matter.

A particular example of this application occurs in chapter III of *Begriffsschrift*. In this chapter, Frege, on the one hand, introduces new letters, such as 'x', 'y', 'z' – which are only interpreted as letters that stand for objects –, or 'F' – which is interpreted as a letter that stands for properties –; and, on the other, he uses the letter 'f' – which in chapter II has been used as a generic unary function letter – as a binary letter that stands for procedures. The new symbols are the minimum requirement in order to justify how objects inherit properties in sequences generated by a procedure. The content of chapter III is, according to Frege – and even from a contemporary point of view –, *pure thought* and can be naturally associated with what we now call "logic". However, this chapter should be distinguished from chapter II, since in the latter we find an example of what we have called "pure concept-script". First, Frege introduces in chapter III symbols that have a fixed interpretation and, second, these symbols acquire a proper meaning, though an abstract one. Hence, although 'x' and 'f' are letters and still express generality, they do not acquire such a variety of readings as an ordinary letter of the concept-script and it is possible to generate atomic formulas – such as '$f(x, y)$' – with these new letters. The formulas obtained

[29]Frege does not rule out the possibility of having nested quantifications in the concept-script, but he explicitly indicates that the nested quantifiers must bound different letters [Frege, 1879, §11, p. 131]. Notice that the occurrence of nested quantifiers of the same variable – as in $\forall x(\phi(x) \to \forall x \psi(x))$ – does not establish a formula as incorrect in contemporary logic.

using the specific symbols of chapter III can be interpreted exactly in the same way as the formulas of contemporary formal languages. In this sense, it can be said that, in chapter III of *Begriffsschrift*, the concept-script is complemented with (applied) logic and therefore is used as the basis for logistics.

The scientific discourses to which the concept-script is adapted determine a specific domain of entities. The proper symbols of this subject matter denote entities in that context, or relations and properties applied to the basic entities. The concept-script does not alter the interpretation of these symbols, but allows that its formulas be interpreted according to the scientific discourse. Hence, the meaning of the adopted terms and the field of application of the concepts of the subject matter indicates how the instances of the quantified letters must be handled in the calculus. In this way, the syntactic conditions of the concept-script are supplemented with semantic ones; only the meaningful expressions of the scientific discourse taken into consideration can be acceptable instances of the letters. Any such discourse allows the intended interpretation of the letters of the concept-script to be specified.

In this context, quantification is handled in the same way as in *Begriffsschrift* – that is, in a purely syntactical way – in spite of the fact that it acquires a meaning it had not before. There is no modification in the use of the quantifiers in the calculus, they are rather interpreted in such a way that semantical restrictions are added to the usual syntactic ones, which we discuss in section 4.1 above. Nevertheless, the possibility of reading the quantifiers in the usual manner in arithmetic – as Frege does, for instance, in several examples in 'Booles rechnende Logik und die Begriffsschrift' [Frege, 1880, pp. 21–27] – does not mean that they are semantically interpreted in the pure concept-script. Quantification of a letter in logistics thus involves considering those expressions that may be put in place of the letter. The difference between quantification in logistics and in pure concept-script is that the instances in the latter are only syntactically determined, while in the former they are also semantically delimited.

5. The nature of the quantification of Begriffsschrift

The final issue we want to address is the possibility of offering an interpretation to the quantifiers of the concept-script that is similar to contemporary ones. This issue is of particular relevance, since there is a common tendency to regard the concept-script of *Begriffsschrift* as a formal system of second-order logic.

5.1. Interpretation of the quantifiers. In accordance with the traditional interpretation of *Begriffsschrift* – shared, for instance, by Peter Sullivan[30] –, it really is quite surprising that Frege had developed as early as 1879 almost

[30] See 'Frege's Logic' [Sullivan, 2005, p. 662].

all those elements that a formal system must have according to contemporary standards of rigour. Up to now, we have tried to show that most of the similarities that lead to such a conclusion to be reached are merely apparent[31]. In what follows, we defend that the usual reading of *Begriffsschrift* is biased because of its submissiveness to a contemporary formal perspective in one particular aspect: the presence of a semantics – in the contemporary sense – for the concept-script. Specifically, scholars such as Beaney claim that Frege offers a semantic interpretation of the quantifiers[32].

Frege does not draw a neat distinction between syntax and semantics. Even if all components of the language are handled exclusively from a syntactical perspective, Frege offers explanations with semantic elements. After all, the author is introducing a completely novel device and, through these semantic remarks, Frege is probably trying to make all the new symbolism he introduces both accessible and understandable to the reader. The three instances of formula (5) we have considered provide evidence that these remarks are not systematic. In spite of his particular exposition, Frege deals with quantification in a purely syntactical way. Our reconstruction in the last section makes this plain: the universal quantifier is a logical symbol whose use in the concept-script is fixed exclusively by certain inference rules. In the presence of a quantifier, the only elements to be considered are those expressions that can be put in place of the quantified letter. We have discussed two syntactical rules that delimit these instances; but there is no trace in *Begriffsschrift* of an account of a general interpretation of quantified letters.

Therefore, in each context of application of the formal resources of the concept-script we should consider a specific range of expressions that could be substituted for a quantified letter. This context of application is closely related to the possibility of using the concept-script as logistics: the complementation of its language with the proper symbols of a given discipline affects the assertibility of the expressions and, hence, the generality of the letters of the concept-script. There are very general contexts of application as well, such as chapter III of *Begriffsschrift*, where there are just a few syntactic conditions that delimit the generality that letters express and, additionally, certain letters – such as 'x' – that have a unique interpretation. We have discussed these assertibility-related aspects in section 4.2.

Now, the range of expressions that can be taken as instances of a quantified letter is almost completely dependent on this context of application. This

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[31]Nevertheless, we will further state elsewhere our position against an interpretation of the concept-script as a second-order formal system. Specifically, all proofs and substitutions in *Begriffsschrift* can be reconstructed coherently with a faithful interpretation of the distinction between function and argument, and the particular theory of quantification developed in this book. One of the main benefits of such a perspective is that there is no need to save Frege from his so-called mistakes: each proof can be explained naturally without constraining it to the tools available in a second-order calculus.

[32]See *The Frege Reader* [Beaney, 1997, pp. 378–379].

should be clear in our discussion concerning formula (5); without a specific formula, the possible instances of a quantified letter – either a function or an argument letter – cannot be fully specified. Furthermore, the fact that the quantified letter is either a function or an argument letter provides no decisive information about the kind of expressions that can be substituted for it. Unlike the variables in a contemporary formal system, the substitution range of the letters in the concept-script is not determined beforehand.

5.2. **The possibility to quantify over functions.** We can discern in the traditional interpretation of *Begriffsschrift* three fundamental claims: argument letters must be considered individual variables; the domain of these variables is the universal class – that is, the collection of all objects in the universe; and the functions of *Begriffsschrift* – which are identified with properties – can be quantified. The first and the third claims entail the thesis that the concept-script of *Begriffsschrift* is a formal system of second-order logic; while the second is part of the common interpretation of Frege's works.

In a substantial part of chapter II of *Begriffsschrift* only the propositional fragment of the concept-script is used[33]. One conclusion of our discussion in section 3.2 is that, regardless of this propositional fragment, in an isolated formula the argument letters of the concept-script cannot be uniformly interpreted and, in particular, they should not be taken to be individual variables. This is our account of the first mentioned claim. In what follows, we will confront the two remaining claims in detail.

In the previous section we deal with Frege's account of generality. According to that account, any symbol in a judgement can be taken to be the argument and then be replaced with a quantified letter. In particular, function letters can be quantified. Many modern commentators – such as Sullivan, Bynum or Beaney[34] – have taken this as an evidence that functions are quantified in *Begriffsschrift* and hence, that the concept-script contains – or should contain – a higher-order quantification.

Let us be clear concerning the account we want to oppose. When the possibility to quantify over functions is defended, it is usually accompanied by an association between Fregean functions and properties, that is, by an attribution of an absolute sense to *Begriffsschrift*'s functions. This view is probably influenced by Frege's position on the matter expressed in *Grundgesetze*. From this point of view, a function letter – seen as the representative of a function in an absolute sense – expresses generality over properties or relations and is therefore interpreted as a predicate variable[35].

[33] Specifically, in the presentation or derivation of Propositions (1) to (51) [Frege, 1879, §§14–19, pp. 137–161].

[34] See 'Frege's Logic' [Sullivan, 2005, p. 667], 'On an Alleged Contradiction lurking in Frege's *Begriffsschrift*' [Bynum, 1973, p. 286] and *The Frege Reader* [Beaney, 1997, p. 76, footnote 52], respectively.

[35] T. W. Bynum makes manifest the association between function letters and functions in 'On an Alleged Contradiction lurking in Frege's *Begriffsschrift*' [Bynum, 1973]. Moreover,

Hence, what is really claimed is that in *Begriffsschrift* there is a quantification over properties, that is, some sort of second-order quantification. However, as we discuss here, this interpretation is not tenable.

Certainly, there are many contexts of application in which the instances of function letters are predicates or conceptual expressions. We are not defending that this is not the case; but that this is not the only kind of contexts in which a function letter can appear. Therefore, we are ultimately arguing against a homogeneous interpretation of function letters in *Begriffsschrift* and, in particular, against the claim that – according to such an interpretation – only predicates can be substituted for function letters.

We have defended, on the one hand, that a function letter has no domain of quantification, but a range of expressions that can be substituted for it; and, on the other, that this range cannot be fully determined beforehand. Even if under particular circumstances – for instance, in a proof – it is possible to state what are the expressions that can take the place of a function letter, those expressions do not necessarily refer to a property. In consequence, function letters do not have a fully specified and general domain of interpretation consisting of properties or relations.

The reading of Frege's account of generality we are considering, according to which functions – in an absolute sense – can be quantified, has led to semantical interpretations of the quantifiers. The most substantive of them can be called "type-neutral" interpretation[36]. According to it, a quantified expression such as:

$$\smile^{\mathfrak{a}}\!\!-\ \Phi(\mathfrak{a}),$$

is ambiguous and, in particular, can be interpreted as expressing that $\Phi(F)$ is the case whatever function – in an absolute sense – $F(\alpha)$ is taken as

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in an editorial note in his translation of *Begriffsschrift*, he renders Frege's phrase "$F(y)$, $F(\mathfrak{a})$ (...) are to be considered different functions of the *argument* F" as "treating $F(y)$ as a function of the *function* F" [Frege, 1972, §27, p. 175, footnote 2, emphasis added].

[36]See, for example, Peter Sullivan's account of some applications of Proposition (58) in 'Frege's Logic':

> "(...) Frege appeals to (direct consequences of) his quantifier axiom [Proposition (58)], $\forall\mathfrak{a}\, f\mathfrak{a} \rightarrow fc$, in justifying second-order inferences, substituting (as we would say) second-level predicates for first-level predicate variables. It would be surely be a misdiagnosis of this to hold that Frege cited a first-order axiom when he needed a second-order one. Rather, his citing the axiom in these cases shows that he did not understand it as first order, but instead as the *type-neutral* principle that if a function holds of every argument it holds of any. It is indeed overwhelmingly the natural view that there are type-neutral logical principles, e.g. that there is a single principle of Barbara, exemplified by properties of any level, to the effect that if wherever one property applies a second does, and wherever the second applies a third does, then wherever the first applies so does the third." [Sullivan, 2005, pp. 672–673, our emphasis]

its argument[37]. Apparently, this type-neutral reading of *Begriffsschrift*'s quantification could also explain the substitutions performed in the example we have considered in section 4.2[38].

We argue in section 3.2 that there is no absolute sense that can be attributed to the notion of *Begriffsschrift*'s function. Accordingly, no hierarchy of levels of functions can be defined in *Begriffsschrift.* In this work, Frege considers properties and objects as ontological categories instead; for instance, throughout chapter III he repeatedly says that certain objects have certain properties. However, there is no evidence in *Begriffsschrift* that he intended properties to have other properties – which would resemble the possibility for a function (in the absolute sense) to be the argument of another function.

According to our reconstruction, the expression '$\Phi(F)$' can be analysed in such a way that 'F' is the argument and '$\Phi(\alpha)$' the function – or, at least, part of it. Now, if '$F(\beta)$' is a function in an absolute sense, say, if it stands for properties, then an absolute sense can be attributed to '$\Phi(\alpha)$' as well. A natural question would then be: which one? Certainly, '$\Phi(\alpha)$' cannot stand for a property, at least in the same sense as '$F(\beta)$'. In order to make sense of this claim, '$\Phi(\alpha)$' cannot stand for a property of the same kind as '$F(\beta)$', that is, there must be a distinction between properties of different levels. However, in *Begriffsschrift* there can be found no such distinction; there simply are no types in this work[39]. Moreover, it is extremely implausible that Frege intended to attribute an absolute sense to the notion of function in the 1879 work and, at the same time, ignored or even did not consider this question. Therefore, the answer to this question is, in one way or another, unacceptable. The ambiguity of a quantified expression – in the sense proposed by the type-neutral interpretation – is thus hardly sustainable. In particular, the reading of the quantifier as involving a quantification over functions – in

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[37]In order to render a function in an absolute sense in this discussion, we use '$F(\alpha)$', where 'α' marks the incompleteness of the function. This use disagrees with our purpose of introducing small Greek letters, but we think that it is clear enough in this particular context.

[38]Taking the substitutions performed in this example – and in all other examples of similar nature – at face value, and recognising that all steps made in these substitutions are valid have persuaded us to rule out an alternative interpretation, according to which the – to the contemporary eyes – oddities of these substitutions are Frege's inaccuracies. See, for instance, T. W. Bynum's account of the substitutions that affect Proposition (60) in the derivation of (93) in his translation of *Begriffsschrift* [Frege, 1972, §28, p. 183, footnote 5].

[39]Departing from a comparative analysis of the presentation of quantification in *Grundgesetze* and in *Begriffsschrift* and, additionally, a remark on the particular character of the notion of function in *Begriffsschrift*, Heck and May come to the same conclusion in 'The Function is Unsaturated':

> "[T]here isn't really a distinction between function and object in *Begriffsschrift*, and accordingly (...) there is no distinction between levels, although there is a distinction between function and *argument.*" [Heck; May, 2013, p. 827].

However, they do not definitely endorse there the claim that a function and an argument are *exclusively* particular components of an expression, which we take to be essential.

an absolute sense – is incompatible with Frege's account of generality in *Begriffsschrift*.

When Frege alludes to the quantification over function letters and substitutes function letters for argument letters, he is not indicating the ambiguity of the quantifier '$-\!\!\overset{\mathfrak{a}}{\smile}\!\!-$', but considering one particular application of its general definition. To say that *Begriffsschrift*'s quantification involves exclusively arguments is completely independent of the nature of the denotation that any suitable argument of certain expression can have. In fact, there is no need to consider a type-neutral interpretation of the quantifiers once we overcome the possibility to attribute an absolute sense to the notion of function. As Frege shows several times, we can take the letter 'F' – which stands for properties in chapter III of *Begriffsschrift* – to be the argument of a suitable function and hence quantify it. In this particular context, there can be no sense of a function – relevant to *Begriffsschrift* – that can be associated with 'F'.

5.3. **Universality in *Begriffsschrift*.** It is common to defend that the quantifiers of the concept-script are not restricted, and that the letters express generality over an absolutely general domain[40]. We will finally discuss this particular claim and defend the idea that there is no general and unrestricted domain of interpretation in *Begriffsschrift*. The presence of a specific range of expressions suitable for taking the place of a quantified letter goes directly against this idea. In fact, it is possible that the substitution range of the same letter does not contain the same expressions in two different settings. By 'setting' we understand a collection of formulas that appear together for some reason – paradigmatically, because they are premisses or assumptions in a proof. The interpretation of the symbols must be homogeneous in order to be consistent with their occurrence in all formulas in a setting. Even if the notion of context of application is more general than that of setting, they are closely related; they both impose conditions that delimit the assertibility of the expressions of the concept-script and, consequently, the generality its letters express. Thus, a formula in a single context of application – for instance, that of arithmetic – can be part of different settings.

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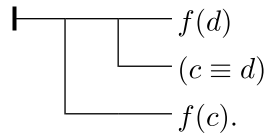
[40]Bynum's account regarding the domain of argument letters in *Begriffsschrift* exemplifies such a standard reading:

> "Arithmetic, according to Frege, is the scientific discipline concerned with numbers; therefore the range of arguments of its functions is the numbers. Maintaining a strict analogy with this view of arithmetic, one can say that, if logic is that scientific discipline concerned with anything at all, then the range of arguments of its functions should be anything at all." [Bynum, 1972, p. 62]

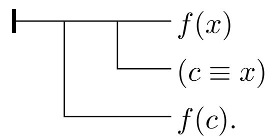
Bynum later affirms that this range is the entire universe, but "*not* a limited "universe of discourse", but *the* universe" [Bynum, 1972, p. 64, footnote 44].

Consider Proposition (53) [Frege, 1879, §21, p. 162]:



(11)

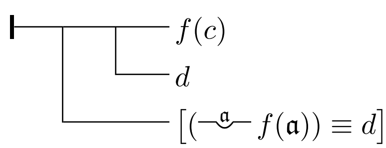
We will exemplify two different settings with the guidance of some proofs in *Begriffsschrift*. The first setting can be found in the derivation of Proposition (92) [Frege, 1879, §28, p. 182], where Proposition (53) – which we have labelled as (11) – is used as one of the premises. In this derivation, Frege does not use (11) as it stands, but performs certain substitutions in it. In particular, 'd' is replaced with 'x', from which we arrive at:



(12)

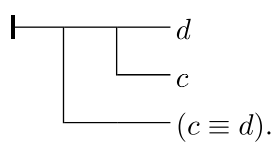
The letter 'x' – as is suggested by its arithmetical use – is interpreted throughout chapter III of *Begriffsschrift* as a letter that stands for objects. This means that 'c' has to be read as a letter that stands for objects as well. Now, in the setting of the derivation of Proposition (92), (11) is interpreted as (12).

As we have discussed, the letters 'c' and 'd' can be propositionally interpreted. Proposition (57), which is very similar to (11), is one of the premises of the derivation of Proposition (68) – which we have labelled as (10) – [Frege, 1879, §22, p. 166]. The second setting is composed of formula (10)[41]:



(10)

and formula (8):



(8)

Leaving aside the fruitfulness of such a setting, since 'c' and 'd' must be uniformly interpreted both in (8) and in (10), the adequate instances of 'c' are only formulas. Therefore, neither 'g' nor '2', to mention just two examples of different kinds, belong to the range of 'c'. This is plainly coherent with reading the formula (11) in the way indicated by (7). Without such a reading, it would not be possible to interpret the transition from (5) to (8). According to modern standards of rigour, these kinds of substitutions entail

[41]We have substituted 'd' for 'b' in (10) for the sake of the example. Clearly, this change is only notational and does not affect the interpretation of (10) at all.

an inconvenient lack of distinction. They are, however, a clear advantage for Frege, for a formula like (11) can be used in settings that have very different logical interpretations.

No instance of 'c' here is an acceptable instance of 'c' in the previous setting: a formula cannot be substituted for 'c' without affecting the assertibility of (12). This circumstance is plainly incompatible with the presence of an absolutely unrestricted domain for 'c'. In fact, as our two examples show, it is even possible that the ranges of a letter in two different settings may be disjoint.

6. Concluding remarks

In spite of what is usually asserted by modern scholars, the formal system of the *Begriffsschrift* is extraordinarily singular. First, the concept-script does not properly have either a language or syntactic rules for the definition of such an important element as the notion of atomic formula. This can be seen as an evidence that the concept-script is devised as a tool for developing a discipline such as arithmetic, so that it expresses the logical relations that link the statements of the discipline and also provides the laws of reasoning needed to rigorously conduct proofs. In fact, Frege did not aim at a formalisation or at a reduction. He confirms this in 'Über den Zweck der Begriffsschrift' [Frege, 1882] considering the nature of the propositions in *Begriffsschrift*:

> "These formulas[42] are actually only empty schemata; and in their application, one must think of whole formulas in the places of A and B – perhaps extended equations, congruences, projections. Then the matter appears completely different." [Frege, 1882, p. 97]

In fact, when the formal elements of the concept-script are applied to express, for instance, arithmetical sentences, the reason for the lack of a proper syntax is patently clear. Atomic formulas are given by the arithmetical language and are organically put in relation with the logical symbols of the concept-script, thereby giving rise to complex formulas. The distinction between function and argument in these formulas stands out as a fundamental tool in this complementation, as it provides a natural and useful way to articulate the acquired symbols and apply to them the formal resources of the concept-script; for instance, quantification.

Second, the concept-script does not have semantics in the contemporary sense of the term. Its quantification theory, based on the division between function and argument, is only syntactically handled. This formal system provides the inference rules that govern the introduction and the elimination of quantifiers in the calculus and some syntactical limitations for the correct application of these rules. There is no semantic interpretation for the letters,

[42]Frege refers to some examples of combinations of conditional strokes and negation strokes, where he uses A and B to mark the place of relevant expressions.

whose generality appears to be of a complete different kind from that of modern variables.

Defending the claim that concept-script of *Begriffsschrift* is a second-order formal system entails contradicting key elements in Frege's exposition. Such a thesis demands, on the one hand, the presence a specific ontological structure that determines both the decomposition in terms of function and argument, and the semantics of a formal system; and, on the other, a complete and anachronistic limitation to the flexibility that characterise the letters of the concept-script. We have argued that there is strong textual evidence in *Begriffsschrift* against the presence of these elements.

We have studied Frege's exposition of the distinction between function and argument, and the notion of generality in *Begriffsschrift* in detail, in order to compare our findings with the traditional reading of this book. However, a comprehensive evaluation of the more general thesis that the concept-script is a second-order formal system would require a complete reconstruction. That would far exceed our aim in this paper, and deserves a proper consideration elsewhere.

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