# *BEGRIFFSSCHRIFT*'S LOGIC

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Abstract. In *Begriffsschrift*, Frege presented a formal system and used it to formulate logical definitions of arithmetical notions and to deduce some noteworthy theorems by means of logical axioms and inference rules. From a contemporary perspective, *Begriffsschrift*'s deductions are, in general, straightforward; it is assumed that all of them can be reproduced in a second-order formal system. Some deductions in this work present—according to this perspective—oddities that have led many scholars to consider it to be Frege's inaccuracies which should be amended.

In this paper, we continue with the analysis of *Begriffsschrift*'s logic undertaken in [1] and argue that its deductive system must not be reconstructed as a second-order calculus. This leads us to argue that *Begriffsschrift*'s deductions do not need any correction but, on the contrary, can be explained in coherence with a global reading of this work and, in particular, with its fundamental distinction between function and argument.

## 1. Introduction

Gottlob Frege developed in *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* [11]—his first major work—a formal system and provided it with a set of basic laws and inference rules. In doing so, he formulated the first formal system in the history of modern logic, which he called '*Begriffsschrift*'.[1] Both the language and syntax of the concept-script are determined by the distinction between function and argument. In fact, the basic laws and the substitutions performed in the calculus take this distinction as their guiding axis.

Too often the—actual or apparent—similarities between Frege's system and some contemporary formal systems have been taken for granted as evidence for a contemporary interpretation of the concept-script.[2] In fact,

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[1]For the sake of clarity, from now on we will use '*Begriffsschrift*' to refer to the book published by Frege in 1879 and 'concept-script' to refer to the formal system developed in it.

Two pages numbers—separated with a semicolon—will be given in quotations taken from Frege's works. The first corresponds to the German edition and the second to the English translation mentioned in the list of references.

[2]See, for instance, Sullivan's general evaluation of *Begriffsschrift* in 'Frege's Logic':

> "From one point of view, Frege's logic *needs* no explanation. The system of logic he presents in *Begriffsschrift* simply *is* modern logic. Furthermore, it needs no subtle or questionable exegesis to recognize it as such: anyone with a basic grounding in contemporary quantificational logic can immediately recognize in *Begriffsschrift* a version of what he

the most common and traditional interpretation of *Begriffsschrift*'s concept-script claims that it consists of a formal language of second-order logic and a deductive system for that language. According to this interpretation, the presentation of the formal language and the deductive system suffers from several inaccuracies. However, advocates of the traditional interpretation defend that a reformulation of the formal language, the basic laws and the inference rules, coupled with the addition of an explicit substitution rule for predicate variables, solves these inaccuracies and results in a formal system by means of which all proofs contained in *Begriffsschrift* can be rendered in a second order language.[3]

We argued in [1] that the conceptual system based on the distinction between function and argument is not compatible with a formal language of second-order logic. In this paper, we offer a detailed analysis of *Begriffsschrift*'s deductive system and justify that it must not be interpreted as a formal system of second-order logic. Specifically, we defend that a reformulation of the calculus of the concept-script in terms of a second-order calculus distorts its nature and, moreover, that some proofs of *Begriffsschrift* are not reproducible by means of this reformulation. We also show that our reading enables the reconstruction of each and every derivation in a way that is compatible with our explanation of the distinction between function and argument.

After this introduction, in the second section we will introduce the components of the language of *Begriffsschrift*'s concept-script. In the third section we will formulate its axiomatic system: that is, the set of basic laws and inference rules. We will then complement this exposition with a reconstruction, in the fourth section, of the substitution processes that take place in *Begriffsschrift*'s deductions. We will take the analysis of a noteworthy derivation done in the fifth section as support for our reconstruction of the concept-script. Finally, in the sixth section, we will evaluate the thesis that the concept-script can be faithfully reconstructed as a second-order formal system.

## 2. Language

The symbols of the language of the concept-script are divided into those that have a fixed meaning and those that express generality [11, §1, p. 1; 111]. This distinction differentiates the logical symbols from what Frege calls 'letters'.

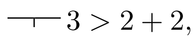
has learned. There are, of course, differences of emphasis, and some points are explained differently from what one would now expect. Some of these divergences are important and we will need below to set them out, and to ask whether they signal important differences in conception of the ground, role or nature of a logical system. Even so, it is remarkable that the differences most likely to cause a modern reader to stumble are wholly trivial matters of notation." [42, p. 661, Sullivan's emphasis]

[3]To illustrate this interpretation, see [10, p. 286], [7, p. 3], [36, p. 101], [42, p. 667], [35, p. 2], [5, p. 12] or [8, p. 356].

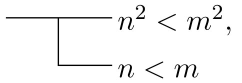
2.1. Frege begins to use letters in the section on Generality of Chapter I of *Begriffsschrift* without distinguishing between different sorts. Letters express generality and have no determinate meaning. In order to ease our exposition, we differentiate between two sorts of letters: *function letters*—such as $f$, $g$ or $h$—and *argument letters*—such as $a$, $b$, $c$ or $d$. Frege only refers explicitly to the former group but this distinction is consistent with his practice.

The concept-script has six different logical symbols: the content stroke, the judgement stroke, the negation stroke, the conditional stroke, the equality symbol and the generality symbol.[4] The content stroke —— indicates that the content of a combination of symbols is taken as a unitary whole. The judgement stroke | serves as an indication of the act of assertion; it marks the affirmation of the content of a statement in a judgement. In *Begriffsschrift*, Frege does not offer an explanation of what he understands by content; for the aim of this paper, it will suffice to assume that the content of a statement corresponds to what it means and that the content of a term is its denotation.

The negation and the conditional concern contents. The negation of $3 > 2 + 2$ is rendered thus:




and the conditional "if $n < m$, then $n^2 < m^2$" is expressed as:

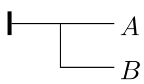



where $n < m$ is the antecedent and $n^2 < m^2$ the consequent.

The equality symbol $\equiv$, unlike the negation stroke and the conditional stroke, relates names [11, §8, pp. 13–14; 124]. For example, '$2^2 \equiv 3 + 1$' expresses that the terms '$2^2$' and '$3 + 1$' have the same content, that is, denote the same object. We will discuss in Section 2.3 two different uses of the equality symbol.[5]

The particular nature of the connectives in *Begriffsschrift* deserves some remarks. Frege presents the conditional in the following way:

> "If $A$ and $B$ stand for assertible contents,[6] there are the following four possibilities:
> (1) $A$ is affirmed and $B$ is affirmed
> (2) $A$ is affirmed and $B$ is denied
> (3) $A$ is denied and $B$ is affirmed
> (4) $A$ is denied and $B$ is denied.
> Now,




[4]On Frege's notation for propositional logic, see [40].

[5]The particular nature of the equality symbol in *Begriffsschrift* has been the object of several analyses. It is far beyond the scope of this paper to consider the difficulties this symbol generates. On this matter, see [34], [27] and [33].

[6]Concerning the notion of assertible content, see Section 4.3 below. For our present purposes, it is enough to assume that a statement or a formula expresses an assertible content.

> stands for the judgement that *the third of these possibilities does not occur, but one of the other three does.*" [11, §5, p. 5; 114–115, Frege's emphasis]

Many contemporary historical studies take this passage as a basis to defend the case that the definition of the connectives in *Begriffsschrift* is truth-functional.[7] On this matter, the first significant aspect is that in 1879 Frege had not yet distinguished between the affirmation of a content and the truth or falsity of that same content. In fact, in the unpublished 'Booles rechnende Logik und die Begriffsschrift' [13, pp. 9ff; 11ff.], Frege states on some occasions that the content of the components of a statement is correct (*richtig*) or incorrect (*falsch*), but his exposition in this work concerning the logical symbols of the concept-script is equivalent to that of *Begriffsschrift*. The distinction between the truth of a content and the acknowledgement of that truth does not appear in Frege's writings until the unfinished 'Logik' [15, p. 2; 2]. In conclusion, the fact that between 1879 and 1882 Frege identifies—or does not neatly distinguish—the acknowledgement of the truth of a content—i.e., its affirmation in a judgement—with the truth of that same content shows that he neither attributed a truth value, in a technical sense, to the components of statements nor defined the connectives in a truth-functional way.[8]

If Frege had introduced the concept of truth-value and offered a truth-functional definition of the connectives in *Begriffsschrift*, he would have shown that the propositional basic laws are universally valid. However, he does not provide such a proof.[9]

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[7]See, for instance, van Heijenoort's introduction to his edition of *Begriffsschrift* [43, p. 1]. See also [41, pp. 76–82] and [42, p. 664]. However, Baker and Hacker [3, pp. 114–119] argue against the presence of truth-functional definitions for the connectives. A further development of Baker's views on this matter can be found in [2].

[8]The first technical use of the concept of truth-value is due to C.S. Peirce (1839–1914). He introduces the notion of truth value as a logical object in his 1885 paper 'On the Algebra of Logic':

> "Let propositions be represented by quantities. Let $\boldsymbol{v}$ and $\boldsymbol{f}$ be two constant values, and let the value of the quantity representing a proposition be $\boldsymbol{v}$ if the proposition is true and be $\boldsymbol{f}$ if the proposition is false." [37, p. 166]

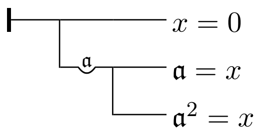
This presentation enables the formulation of a truth-functional definition of the conditional:

> "A proposition of the form
>
> $$x \multimap\!\!< y$$
>
> is true if $x = \boldsymbol{f}$ or $y = \boldsymbol{v}$. It is only false if $y = \boldsymbol{f}$ and $x = \boldsymbol{v}$." [37, p. 181]

Compare Peirce's introduction of the notion of truth-value and his definition of the conditional with what Frege says about this connective.

[9]As an introductory step to Chapter II of *Begriffsschrift*, Frege justifies the evidence of the first basic laws and some propositions [11, §§14–19, pp. 26–50; 137–161]. He intends to clarify the meaning of these basic laws and the notation used in derivations but he does not have a systematic aim in mind. In Frege's words: "[t]he following derivation would tire the reader if he were to trace it in all its details. Its purpose is only to keep in readiness the answer to any question about the derivation of the law" [11, §13, p. 26; 137]. In fact, on many occasions Frege's informal clarifications even adapt the logical form of these basic laws to ease his explanation.

2.2. The generality symbol is composed of two different elements: a concavity placed in the middle of a content stroke—not necessarily attached to a judgement stroke—and a German letter. In *Begriffsschrift*, Frege employs the German letters $\mathfrak{a}, \mathfrak{d}, \mathfrak{e}$ and $\mathfrak{F}$: we will refer to the first three as argument German letters and to $\mathfrak{F}$ as a function German letter. In Frege's words, the generality symbol "*delimits the scope of the generality signified by this letter*" [11, §11, p. 20; 131, Frege's emphasis]. Accordingly, the position of the concavity indicates its scope and the German font points out which letter is bound by the concavity. For instance, the expression:

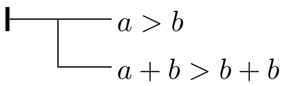



stands for the judgement: "if each square root of $x$ is $x$ itself, then $x = 0$".[10]

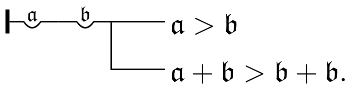
In *Begriffsschrift*, the notion of generality is deeply linked with the distinction between function and argument, which is acknowledged as the basic element of the concept-script. Frege introduces the distinction between function and argument as a particular decomposition of significative expressions. An expression can be decomposed into two components: a variable part, the *argument*, which is the component that can be replaced, and a fixed part, the *function* [11, §9, p. 15; 126]. The two main characteristics of this distinction are the following: on the one hand, it is not unique, i.e., it can be made in different ways; and, on the other, it does not obey any pre-established guideline: any component can be the argument or the function. As a result, a single expression can be analyzed in different ways in the course of a derivation.

According to Frege's proposal, a symbol in the expression of a judgement—either a function letter or an argument letter—is taken to be the argument and can be replaced with a German letter [11, §11, p. 19; 130]. In fact, he explicitly states that the function is what remains when a symbol in a statement is regarded as the argument and a German letter replaces it.

All letters express generality. When it is convenient to delimit their scope, German letters can be used; as we have said, their scope is determined by the position of the corresponding concavity. In contrast, the scope of italic letters encompasses the totality of the judgement where they occur. In Frege's words, "*An italic letter is always to have as its scope the content of the whole judgement*, and this need not be signified by a concavity in the content stroke" [11, §11, p. 21; 131-132, Frege's emphasis]. Note that the following expression:




stands for the same judgement as:



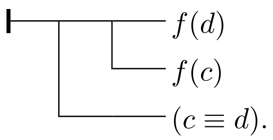

[10] This example and its translation into natural language are due to Frege. See [14, p. 105; 99].

2.3. *Begriffsschrift*'s division into chapters reflects the structure of its content. Chapter I is devoted to the presentation of the language and the elements related to it, such as the distinction between function and argument. In contrast, Chapters II and III almost exclusively contain deductions. The introduction of the symbolism is thus isolated from the presentation and application of the axiomatic system.

Chapter II contains the exposition of the basic laws of the concept-script and the deduction of those logical theorems which are needed in the proofs of Chapter III. The only non-logical symbols occurring in the propositions of Chapter II are letters. We claim that the propositions contained in this chapter do not even have a definite meaning but, on the contrary, can be read in different ways and hence be applied to a variety of possible applications.

In support of this claim, we evaluate a significant example of how the propositions of Chapter II are interpreted by Frege. For this matter, consider Proposition (52):

Pr. (52)




(1)

This is a basic law of the concept-script and will be labeled **L7** in the next section. It has a unique interpretation in the concept-script: the case in which $c$ and $d$ denote the same content, $f(c)$ is affirmed and $f(d)$ is denied does not occur. However, from the point of view of contemporary logic, this law admits at least two different and incompatible interpretations depending on whether $c$ and $d$ represent individual terms or formulas. We need to use a contemporary formal language in order to distinguish between these two interpretations.

First, the letters $c$ and $d$ can be seen as individual variables and $f$ as a metavariable for a formula:

$$x = y \to (\phi(x) \to \phi(y)). \tag{2}$$

A particular case of this reading consists of taking $f$ as a predicate variable:

$$x = y \to (Xx \to Xy). \tag{2'}$$

Second, $c$ and $d$ can be interpreted as propositional variables. Hence, $f(c)$ would be any formula in which $c$ occurs. It has no equivalent in the formal language. Accordingly, (1) could be rendered in the meta-language as follows:

If $\phi$ and $\psi$ have the same content, then if $\Phi(\phi)$ then $\Phi(\psi)$.

This could be expressed by means of the following meta-theorem:

$$(\phi \leftrightarrow \psi) \to (\Phi(\phi) \to \Phi(\psi)). \tag{3}$$

Note that the equality symbol occurring in (1) can be read in two different ways, depending on the interpretation of the letters $c$, $d$ and $f$. If $c$ and $d$ are interpreted as individual variables, then $\equiv$ has to be interpreted as an equality symbol. If, on the contrary, $c$ and $d$ are read as formulas, then $\equiv$ is a biconditional.[11]

It is fundamental to take into account that Proposition (52), i.e. formula (1), cannot be linked beforehand to any of its possible interpretations. The same happens with the basic law $(c \equiv c)$—Proposition (54)—and with Propositions (53) and (55)–(57).[12] In fact, a given interpretation of a letter of the concept-script might be admissible in one derivation and inadmissible in another. On some occasions, Proposition (52) is used in a propositional way—that is, according to reading (3)—and on other occasions Frege employs Proposition (53)—which is logically equivalent to (52)—in a way coherent with the reading expressed in (2).[13]

All in all, this example shows that there can be no definition of atomic formula in the concept-script as it is presented in Chapter II. The minimal expression of this language, $f(a)$ can be interpreted in different ways, but these can only be determined in the particular context or derivation in which they occur. In fact, $f$ and $a$ are simply letters; there are no non-logical constants in pure concept-script that could help us determine a particular interpretation of the formulas where $f(a)$ occurs. The lack of such a fundamental element as the definition of an atomic formula is a clear indication that the language of the concept-script is not a formal language in the contemporary sense. It should rather be seen as a structural language, in the sense that its formulas only express either the kind of relations we render in logic as in (2), or relations we would need to render in the meta-language as in (3).[14]

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[11] We offer a more detailed discussion concerning formula (1) and, in particular, its different interpretations, in [1, pp. 325–327].

[12] Pr. (53) $f(c) \rightarrow ((c \equiv d) \rightarrow f(d))$
Pr. (55) $(c \equiv d) \rightarrow (d \equiv c)$
Pr. (56) $((d \equiv c) \rightarrow (f(d) \rightarrow f(c))) \rightarrow ((c \equiv d) \rightarrow (f(d) \rightarrow f(c)))$
Pr. (57) $(c \equiv d) \rightarrow (f(d) \rightarrow f(c))$

For the sake of clarity and economy, here and henceforth we formulate all concept-script propositions that are used in our discussion in a hybrid notation. The only exception will be the formulas appearing in quotes. All judgement and content strokes will be eliminated and all concavities and connectives will be rendered according to their contemporary equivalents, while the different typefaces used by Frege will be maintained.

This reformulation of the concept-script propositions comes at the cost of omitting, first, that the concept-script is a calculus of judgments and, second, the epistemological role that—according to Frege—judgments play in reasoning. A global analysis of the concept-script should take these circumstances into account. That said, since in this paper we focus on the formal aspects of the *Begriffsschrift* calculus, we decided to prioritize notational simplicity.

[13] The propositional uses of Pr. (52) in *Begriffsschrift* can be found in the derivations of Prs. (75), (89) and (105). Frege uses Pr. (53) according to reading (2) in the derivation of Pr. (92).

In this footnote and throughout this paper we often refer to specific propositions of *Begriffsschrift*. Sometimes, we refer to a proposition without formulating it; on those occasions, our discussion does not depend on the specific content of that proposition.

[14] This is thoroughly coherent with Frege's use of the concept-script in Chapter II of *Begriffsschrift*. In this chapter, Frege presents the formal resources he needs in the derivations of Chapter III. In this last chapter, the purely logical propositions of Chapter II are not used as they are deduced, since Frege performs certain substitutions in order to

The goal of Chapter III of *Begriffsschrift* is to offer proofs of some propositions about sequences whose justification was thought to require an appeal to intuition.[15] In this chapter Frege introduces new letters, such as $x$, $y$, $z$—which are interpreted as letters that stand for objects—or $F$—which is interpreted as a letter that stands for properties. Moreover, he uses the letter $f$—which he has been using as a generic unary function letter in Chapter II—as a binary letter that stands for, as Frege puts it, procedures, that is, as a parameter of a rule that, once applied to an object of a particular range, returns one or more objects of the same range. The successor operation or the father-son relation are examples of procedures. Therefore, $f(x, y)$ and $F(x)$, if they occur in Chapter III, have to be seen as atomic formulas.

The introduction of new symbols[16] goes along with logical definitions of arithmetical notions: that is, hereditary property, weak and strong ancestral, and function in a mathematical sense. These notions contribute to conferring a definite—yet abstract—meaning to the propositions of Chapter III. As a result, the use of new letters implies that the propositions obtained using these specific symbols can be interpreted exactly in the same way as the formulas of contemporary formal languages. For instance, Frege introduces the definition of the notion of hereditary property in Proposition (69):[17]

$$\text{Pr. (69)} \qquad [\forall\mathfrak{d}(F(\mathfrak{d}) \to \forall\mathfrak{a}(f(\mathfrak{d},\mathfrak{a}) \to F(\mathfrak{a})))] \equiv \text{Her}(F).$$

and provides the following reading in natural language:

> "*If from the proposition that* $\mathfrak{d}$ *has the property* $F$*, whatever* $\mathfrak{d}$ *may be, it can always be inferred that each result of an application of the procedure* $f$ *to* $\mathfrak{d}$ *has the property* $F$*,*
>
> *then I say:*
>
> *'The property* $F$ *is hereditary in the* $f$*-sequence'*." [11, §24, p. 58; 170, Frege's emphasis]

## 3. Logical system

It is a commonplace to mention that *Begriffsschrift*'s concept-script is the first formal system in the history of logic. This means that, in addition to a regimented language, it has a clearly distinguishable set of basic laws and a set of inference rules.

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employ them as premises. These substitutions determine a specific interpretation of the formulas.

[15]On these proofs' independence from intuition, see [6, p. 333].

[16]Recall that we take the binary function letter $f$, which is a letter that stands for procedures, as a new symbol.

[17]We follow, save minor details, the notational conventions adopted by Boolos [6, p. 332] in our reformulation of *Begriffsschrift* propositions. Hence, we use Her($F$) as a substitute for the Fregean symbol:

$$\begin{array}{ll} \delta & F(\alpha) \\ \big| & \big( \\ \alpha & f(\delta,\alpha). \end{array}$$

which expresses that $F$ is a hereditary property.

3.1. Frege grants a special status to some propositions of *Begriffsschrift*, which are designated as 'basic' or 'fundamental laws' (*Grundgesetze*), or 'primitive laws' (*Urgesetze* or *ursprüngliche Sätze*).[18] These *basic laws* can be divided into four groups: the first six belong to propositional logic, the following two concern equality and the last is devoted to generality:[19]

**L1:** $a \to (b \to a)$.
**L2:** $(c \to (b \to a)) \to ((c \to b) \to (c \to a))$.
**L3:** $(d \to (b \to a)) \to (b \to (d \to a))$.
**L4:** $(b \to a) \to (\neg a \to \neg b)$.
**L5:** $\neg\neg a \to a$.
**L6:** $a \to \neg\neg a$.
**L7:** $(c \equiv d) \to (f(c) \to f(d))$.
**L8:** $(c \equiv c)$.
**L9:** $\forall \mathfrak{a} f(\mathfrak{a}) \to f(c)$, for any argument $c$.

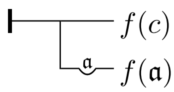
The multiple readings applicable to a single proposition bear witness to the plasticity of the letters of the concept-script. This is especially visible in the basic laws **L7** and **L8**. We have indicated in the previous section the possible interpretations of basic law **L7**; clearly, **L8** can have similar readings.

The basic law **L9** is more general than what a superficial reading would suggest. Actually, following Frege's parlance, **L9** can be read as follows:

> It cannot be affirmed that $f(\mathfrak{a})$ is a fact whatever argument can take the place of $\mathfrak{a}$ and, at the same time, denied that, for any appropriate argument $c$, $f(c)$ is a fact.

This reading is perfectly coherent with Frege's explanations after the introduction of basic law **L9**:

Pr. (58)



**L9**

> "$\overset{\mathfrak{a}}{\smile}$ $f(\mathfrak{a})$ means that $f(\mathfrak{a})$ occurs whatever we may understand by $\mathfrak{a}$. Therefore, if $\overset{\mathfrak{a}}{\smile}$ $f(\mathfrak{a})$ is affirmed, $f(c)$ cannot be denied. This is what our sentence expresses." [11, §22, p. 51; 162]

In fact, these explanations are a natural extension of Frege's reading of quantification:

[18]In *Begriffsschrift*, the basic laws only receive that name—*Grundgesetze*—in the table of contents [19, p. xvi]. Frege does not ever call them 'axioms'. In fact, Frege's use in all of his works written soon before or after 1879 shows that, on the one hand, he employs the term '*Axiom*' only to refer to the axioms of arithmetic or geometry; and on the other hand, he clearly distinguishes this terminology from that which he employs to designate the basic laws of logic (see [12, p. 50] and Frege's letter to Anton Marty dated August 29, 1882 [23, p. 163]).

[19]See the correspondence between the basic laws listed here and their number label in *Begriffsschrift*:

| | | | | | |
|---|---|---|---|---|---|
| **L1** | Pr. (1) | **L4** | Pr. (28) | **L7** | Pr. (52) |
| **L2** | Pr. (2) | **L5** | Pr. (31) | **L8** | Pr. (54) |
| **L3** | Pr. (8) | **L6** | Pr. (41) | **L9** | Pr. (58) |

> "The horizontal stroke situated the left of the concavity in
>
> $$\vdash\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})$$
>
> is the content stroke of the circumstance that $\Phi(\mathfrak{a})$ holds, whatever we may put in the place of $\mathfrak{a}$. The horizontal stroke to the right of the concavity is the content stroke of $\Phi(\mathfrak{a})$, and here we must think of $\mathfrak{a}$ as replaced by something definite." [11, §11, pp. 19–20; 130]

As we said earlier, the concept-script lacks non-logical constants. Letters are the only non-logical symbols at its disposal. There is thus no doubt that the letter $c$ occurring in the consequent of **L9** is a letter in Frege's sense. However, taking the expressive limitations of the concept-script into account, the author's intentions about the inclusion of this basic law have to be considered: Frege intends to reproduce the *dictum de omni*, according to which if a function is a fact for all arguments, then it is a fact for any argument. This is why we have included the clause "for any argument $c$", which is absent in the text.[20]

In basic law **L9**, the letters $\mathfrak{a}$ and $c$ indicate the argument place. In this sense, they specify the syntactic role of some symbols occurring in the expression, but they do not determine their semantic nature. Accordingly, **L9** can also be read as if function letters were included in place of $\mathfrak{a}$ and $c$. We will discuss two applications of this basic law which emphasize this fundamental circumstance in Sections 3.2 and 5.1. The substitution of function letters for the argument letters occurring in **L9** is important considering the interpretation of this basic law, but it does not imply a need for formulating an alternative basic law.[21] It does, however, witness the singularity of the concept-script; the replacement of argument letters with function letters shows that this formal system does not contain a formal language in the contemporary sense nor is it presented as a second-order formal system.

3.2. The concept-script has a set of inference rules that govern the deduction of new propositions. Frege declares in *Begriffsschrift* that Modus Ponens is the only inference rule of the concept-script [11, §6, pp. 9–10; 119–120] and he supports this claim by citing methodological reasons. It is, for sure, the only inference rule of the concept-script that permits the deduction of a proposition from several other propositions. However, in the presentation of the generality symbol, the author introduces two other inference rules without making explicit that they are so; these additional rules permit obtaining a

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[20] In this connection, it is illustrating to compare Frege's mode of presentation of basic law **L9** with the introduction of an equivalent law made by D. Hilbert (1862–1943) and W. Ackerman (1896–1962) in *Grundzüge der Teoretischen Logik*:

> "Dazu kommen jetzt als zweite Gruppe zwei *formale Axiome* für „*alle*",
> *und* „*es gibt*" hinzu:
> e) $(x)F(x) \to F(y)$.
> f) $F(y) \to (Ex)F(x)$.
> Das erste dieser Axiome bedeutet: „Wenn ein Prädikat $F$ auf alle $x$ zutrifft, so trifft es auch auf ein beliebiges $y$ zu"." [32, III, §5, p. 53, authors' emphasis]

[21] For a discussion about the possible interpretations of basic law **L9**, see [1, pp. 335–337].

new proposition from a single proposition. In short, Frege mentions three different rules, to which we will refer as Modus Ponens, Generalization and Confinement of the Quantifier. It is disputable whether the concept-script requires a Substitution Rule; in the following section we will discuss this matter in detail.

In order to ease the exposition of the inference rules, we use $A$ and $B$ as parameters for formulas, and $\Phi(a)$ and $\Phi(f)$ as expressions where the letters $a$ and $f$ occur, respectively. Since $a$ and $f$ occur in $\Phi(a)$ and $\Phi(f)$, they can be taken to be their argument; some or all occurrences of $a$ in $\Phi(a)$ and of $f$ in $\Phi(f)$ are argument places, respectively. Moreover, we refer to the generality symbol as the 'universal quantifier'.

*Modus Ponens* (hereinafter, "[MP]") is the main inference method of the concept-script. According to it, one can infer the consequent of a conditional from this conditional and its antecedent. Frege introduces it as follows [11, §6, pp. 7–8; 117]:

$$\frac{\begin{array}{l} A \to B \\ A \end{array}}{B.}\ [\text{MP}]$$

*Generalization* (hereinafter, "[G]") is the first inference rule introduced by Frege that affects the generality symbol. It permits the deduction of a proposition that results from the replacement of an italic letter with a German letter and the addition of a universal quantifier in a given proposition. Following Frege's guidelines, it can be formulated thus [11, §11, p. 21; 132]:

$$\frac{\Phi(a)}{\forall \mathfrak{a}\, \Phi(\mathfrak{a}),}\ [\text{G}]$$

if $a$ only occurs in the argument places of $\Phi(a)$.

The restriction over $a$ forces all occurrences of $a$ in $\Phi(a)$ to be considered argument places and thus can be rephrased as follows: if there is no occurrence of $a$ in $\forall \mathfrak{a}\, \Phi(\mathfrak{a})$. From a contemporary perspective, it is natural to see, in this context, $\Phi(\mathfrak{a})$ as the result of replacing each occurrence of $a$ in $\Phi(a)$ with $\mathfrak{a}$; the restriction on $\Phi(a)$ is then observed. We maintain this reasoning in the presentation of the following inference rules.

The formulation of [G] concerns argument letters. However, this inference rule can be applied to function letters as well; it is enough to avoid any occurrence of the relevant italic letter in the resulting proposition:

$$\frac{\Phi(f)}{\forall \mathfrak{F}\, \Phi(\mathfrak{F}),}\ [\text{G}]$$

if $f$ only occurs in the argument places of $\Phi(f)$, i.e., if $f$ does not occur in $\forall \mathfrak{F}\, \Phi(\mathfrak{F})$.

In *Begriffsschrift*, all uses of [G] are complemented with an application of the Confinement of the Quantifier.[22] We have already stated that Frege does not explicitly indicate the use of these two inference rules. In fact, given the conventional use he intends for italic letters, [G] can be seen as a

[22]See the derivations of Prs. (97) and (109).

purely notational device; and, precisely for this reason, this inference rule is significant in the reconstruction of substitutions.

Finally, the *Confinement of the Quantifier* (hereinafter, "[C]") is similar to the rule of Generalization, but it permits instead the addition of a universal quantifier in the consequent of a conditional. Frege introduces [C] as follows [11, §11, p. 21; 132]:

$$\frac{A \rightarrow \Phi(a)}{A \rightarrow \forall \mathfrak{a}\, \Phi(\mathfrak{a}),}\ [\mathrm{C}]$$

if $a$ does not occur in $A$ and only occurs in the argument places of $\Phi(a)$, i.e., if $a$ does not occur either in $A$ or in $\forall \mathfrak{a}\, \Phi(\mathfrak{a})$.

In a similar way to [G], Frege presents [C] using argument letters. However, this same formulation could be applied to function letters: a function letter can be taken to be the argument of an expression and thus be quantified.

In order to shorten some deductions, Frege formulates a derived rule of [C] [11, §11, p. 22; 133], which we will call [C′]:

$$\frac{B \rightarrow (A \rightarrow \Phi(a))}{B \rightarrow (A \rightarrow \forall \mathfrak{a}\, \Phi(\mathfrak{a})),}\ [\mathrm{C}']$$

if $a$ does not occur in $A$ or in $B$ and only occurs in the argument places of $\Phi(a)$, i.e., if $a$ does not occur either in $A$, $B$ or in $\forall \mathfrak{a}\, \Phi(\mathfrak{a})$.

The author does not present [C′] as a derived rule and, in fact, even if he informally justifies it, he does not offer a rigorous proof.

## 4. Substitutions

Every proof performed in Chapters II and III of *Begriffsschrift* involves two propositions from which, through the available inference rules, a new proposition is obtained. Frege does not usually employ in the derivations the propositions needed as premises exactly in the same form in which they have been proved: the premises in a proof are subject to certain substitutions. However, Frege seldom makes explicit the result of performing substitutions in the premises and he never comments upon any particular replacement. In fact, the author does not provide a detailed explanation of every kind of substitution performed in proofs nor does he introduce a rule of inference in order to handle them in the calculus. Frege limits himself to offering a generic remark about their nature.

We provide in this section a thorough analysis of the substitutions performed in the proofs of *Begriffsschrift*. According to this analysis, some of the substitutions proposed by Frege which involve function letters cannot be reconstructed as substitutions of complex expressions for predicate variables. The function letters to be replaced on those occasions cannot be interpreted as letters that stand for properties and, therefore, these substitutions are not reproducible in a second-order formal system. However, as our analysis shows, all substitutions mentioned in *Begriffsschrift*—and specifically those that are not reconstructible by means of second-order logic—are natural from the point of view of a logic based on the function-argument scheme.

Even though Frege does not distinguish between different kinds of substitutions, from a contemporary perspective it is convenient to do so. In

*Begriffsschrift* there can be found three different kinds of substitution: propositional replacements (analyzed in Section 4.1), alphabetic replacements (Section 4.2) and substitutions of appropriate expressions for italic letters (Section 4.3).

4.1. The concept-script possesses a propositional fragment: the Propositions (1)–(51) can be only propositionally interpreted and their proof requires exclusively basic laws **L1**–**L6**. Frege does not isolate this fragment nor does he show any intention of viewing it in this way. However, for our present purposes, it is convenient to consider this set of propositions—along with [MP]—as the propositional fragment of the concept-script. Note that it can be extracted from the concept-script in a natural way.

All substitutions performed in the propositional fragment of *Begriffsschrift* consist in the replacement of all occurrences of an argument letter with a formula or, in particular, with another argument letter. Since there is no quantification in this fragment, all these substitutions are trivial.

4.2. Frege briefly comments on alphabetic replacements as a particular kind of substitution. They always involve the substitution of German letters:

> "Naturally, it is permitted to replace one German letter throughout its scope by another particular one provided that there are still different letters standing where different letters stood before. This has no effect on the content." [11, §11, p. 21; 131]

The replacement of a German letter with another of the same kind does not affect the meaning of the proposition where it is performed; in fact, it is introduced, in general, for pragmatic reasons, namely, in order to avoid conflicts of quantification.

According to a contemporary exposition, an alphabetic replacement in a formula $A$ consists in the replacement of a subformula of $A$ of the form $\forall\mathfrak{a}\Phi(\mathfrak{a})$ with the formula $\forall\mathfrak{e}\Phi(\mathfrak{e})$. Note that the resulting formula is logically equivalent to $A$.[23]

We have already commented that propositional replacements are trivial when they are performed in the propositional fragment of the concept-script. In the remaining proofs of *Begriffsschrift*, on some occasions Frege proposes substitutions of formulas for letters that require the performance of alphabetic replacements. In this context, conflicts of quantification can occur. Frege systematically avoids them, but offers no explanation in this regard. He seems to follow the implicit rule according to which if there are conflicts of quantification, then it is enough to perform the alphabetic replacements needed to avoid them.

An example will help to enlighten this strategy. Let us assume that we wish to substitute $\forall\mathfrak{e}(c \to f(\mathfrak{e}))$ for $a$ in the following theorem of the

[23]Frege performs alphabetic replacements in the derivation of the following propositions of *Begriffsschrift*: §25, Pr. (70); §27, Pr. (77); §28, Pr. (93); §31, Prs. (116) and (118).

concept-script:[24]

$$\forall\mathfrak{e}[(f(\mathfrak{e}) \to a) \to (\forall\mathfrak{a} f(\mathfrak{a}) \to a)]. \tag{4}$$

This substitution is not possible, since the resulting expression would not be a well-formed formula of the concept-script. Note that the quantifier $\forall\mathfrak{e}$ occurring in the instance of $\mathfrak{a}$ would be in the scope of the quantifier $\forall\mathfrak{e}$ occurring in (4), and this kind of overlapping is not acceptable in the concept-script [11, §11, p. 21; 131]. Hence, before performing the purported substitution, it is necessary to apply an alphabetic replacement in (4), for instance, the replacement of $\mathfrak{e}$ with $\mathfrak{b}$:

$$\forall\mathfrak{b}[(f(\mathfrak{b}) \to a) \to (\forall\mathfrak{a} f(\mathfrak{a}) \to a)],$$

The substitution of $\forall\mathfrak{e}(c \to f(\mathfrak{e}))$ for $a$ is now possible. The result is the following formula:

$$\forall\mathfrak{b}[(f(\mathfrak{b}) \to \forall\mathfrak{e}(c \to f(\mathfrak{e}))) \to (\forall\mathfrak{a} f(\mathfrak{a}) \to \forall\mathfrak{e}(c \to f(\mathfrak{e})))].$$

The replacement of an argument German letter $\mathfrak{a}$ with a function German letter $\mathfrak{F}$ is one particularly noteworthy specification, although it is not properly an alphabetic replacement.[25]. In *Begriffsschrift* Frege proposes this substitution only when the appropriate instances of $\mathfrak{a}$ are functional expressions[26] This replacement restricts the possible interpretations of the initial formula and thus it cannot be said to be an alphabetic replacement. However, the replacement mechanism involved in the change of $\mathfrak{a}$ with $\mathfrak{F}$ is essentially the same as in the case of alphabetic replacements.

4.3. The third kind of replacement encompasses any other substitution that takes place in the derivations of *Begriffsschrift*. Unlike alphabetic replacements, the result of performing this kind of substitution is not equivalent, in general, to the initial formula. This substitution consists in the replacement of the particular argument of a formula; in other words, it consists in the replacement of an italic letter in all its occurrences with an appropriate instance. Being an element of the derivations, the substitutions we are about to consider occur in the setting determined by the deduction of a formula which results from performing specific replacements to an original formula.

Alphabetic replacements affect only those subformulas that correspond to the scope of the quantifier of the replaced German letter, whereas any other kind of substitution involves the whole formula in which it is performed, that is, the totality of the scope of the letter that is to be replaced. This emphasizes two different applications of the generality the letters of the

[24]This formula results from a single application of [G] to Proposition (61):

Pr. (61) $$(f(c) \to a) \to (\forall\mathfrak{a} f(\mathfrak{a}) \to a)$$

[25]In *Begriffsschrift* applications of the replacement of $\mathfrak{a}$ with $\mathfrak{F}$ can be found in the derivations of Prs. (77) and (93).

[26]From now on, 'functional expression' will refer to an appropriate instance of a function letter. Following Frege's practice in Chapter III of *Begriffsschrift*, in order to render the circumstance that function letters have an arity—which must be preserved by their instances—we employ the capital Greek letter $\Gamma$. For instance, $g(\Gamma, a) \to h(\Gamma)$ is a functional expression and can be an appropriate instance of $f(\Gamma)$. The use of $\Gamma$ will be especially useful in substitutions, since it can help to indicate how a particular instance of a function letter must fit in the corresponding function.

concept-script express. In order to make this blatant, Frege uses different typefaces:

> "*Other substitutions are permitted only if the concavity follows immediately after the judgement stroke* so that the content of the whole judgement constitutes the scope of the German letter. Since, accordingly, this is a specially important case, we will introduce the following abbreviation for it: *An italic letter is always to have as its scope the content of the whole judgement*, and this need not be signified by a concavity in the content stroke." [11, §11, p. 21; 131–132, Frege's emphasis]

The substitution for an italic letter relies on the fact that the replaced italic letter, being the replaceable component of an expression, is taken to be the argument of this expression. Since in the course of a proof a single proposition can be subject to different substitutions, it is clear that its analysis changes in each step of the substitutions: each italic letter that is substituted is taken to be the argument, so that we view the whole proposition in different ways according to the substitutions to be performed.

The specification of what is an appropriate instance of a letter that is about to be substituted leads to the notion of assertibility. In the presentation of the judgement stroke, Frege distinguishes between assertible (*beurtheilbare*) and unassertible (*unbeurtheilbare*) contents [11, §2, p. 2; 112]. His example of an unassertible content is the content of 'house'. In contrast, the content of all statements can be asserted and thus be subject to the act of affirmation in a judgement.

In the context of substitutions, the relevant circumstance regarding assertibility is the preservation of the assertibility of a proposition after a substitution. Frege hints at this when he deals with the generality expressed by a German letter:

> "The meaning of a German letter is subject only to the obvious restrictions that [1] the assertibility (§2) of a combination of symbols following the content stroke must remain intact, and [2] if the German letter appears as a function symbol, this circumstance must be taken into account." [11, §11, p. 19; 130]

As will be clear in what follows, we take a substitution as a process of instantiation; all letters express generality and the substitution of an expression for a letter consists in picking one instance from those possible. Frege suggests that the generality of a letter—leaving aside the circumstance that the letter be functional—is limited only by the assertibility of the expression in which it occurs.

However, Frege does not develop further specific conditions for the assertibility of an expression. There seem to be certain syntactic guidelines, such as the correct formation of complex formulas or the use of German letters inside the scope of a concavity. Nevertheless, the nature of concept-script letters prevents a precise definition of assertibility; only when given a particular pair of formulas $A$ and $B$ and a letter $a$ occurring in $A$ is it possible to determine whether $B$ can be substituted for $a$ in $A$ without affecting the assertibility of $A$. This means that a general and unitary specification of an appropriate instance of a letter, even for argument or function letters as a whole, is not possible.

Consider, for example, the derivation of Proposition (68):

Pr. (68) $$(\forall \mathfrak{a} f(\mathfrak{a}) \equiv b) \to (b \to f(c)).$$

The premises in this proof are the following:

Pr. (57) $$(c \equiv d) \to (f(d) \to f(c)). \tag{5}$$

Pr. (67) $$[(\forall \mathfrak{a} f(\mathfrak{a}) \equiv b) \to (b \to \forall \mathfrak{a} f(\mathfrak{a}))] \to [(\forall \mathfrak{a} f(\mathfrak{a}) \equiv b) \to (b \to f(c))]. \tag{6}$$

Note that the consequent of (6) corresponds to Proposition (68). Frege's procedure in this proof consists in obtaining an instance of (5) that coincides with the antecedent of (6). Then it is possible to apply [MP] to (6) and the instance of (5) and thus conclude Proposition (68) as a result. Accordingly, among the replacements that affect formula (5), Frege indicates that $b$ is to be substituted for $d$ in (5). The result of this substitution is the following:

$$(c \equiv b) \to (f(b) \to f(c)) \tag{7}$$

Since (6) and (7) are part of the same derivation, the letters have to be interpreted coherently in both formulas. Clearly, $b$ can only be propositionally interpreted in (6) and this extends to its occurrences in (7). Therefore, a further restriction is imposed upon the occurrences of $c$, for it occurs in (7) in the subformula $(c \equiv b)$. As a result, the only appropriate instances for $c$ and $b$ in this derivation are expressions for assertible contents, namely, formulas. Having said that, it is clear that if (5) and (6) are taken in isolation, the letters $c$ and $d$ can also be interpreted as individual variables (see Section 2.3).[27]

Substitutions of appropriate expressions for italic letters can be significantly heterogeneous. In order to ease our exposition, we will distinguish between cases.

4.3.1. The first case consists in the substitution of one italic letter for another. The instance can be a letter of the same kind, such as in the substitutions of $a$ for $c$ or of $f$ for $g$—provided that $f$ and $g$ have the same arity. These substitutions do not in general involve any modification of the meaning of the proposition in which they are performed. In some sense, they are merely notational changes which, like alphabetic changes, meet a circumstantial need. In this case, as we have already noted, the result of replacing one italic letter with another preserves the equivalence to the original formula.

Even if Frege performs these substitutions directly, strictly speaking they should be carried out as specific stages in the calculus. In fact, a general proof in the concept-script of the replacement of one italic letter with another of the same kind can be provided. The starting point is a simple expression such as $f(a)$, in which $c$ is to be substituted for $a$. Hence:

[27]Frege's explicit concerns about the assertibility of concept-script expressions in his exposition of the notion of generality and the delimitation of the generality of a letter we have just considered show that, in contrast with what Heck and May defend in [30, p. 829], argument letters cannot be *freely* substituted for function letters and vice versa.

(1) $f(a)$.
(2) $\forall\mathfrak{a} f(\mathfrak{a})$, [G]: (1).
(3) $\forall\mathfrak{a} f(\mathfrak{a}) \rightarrow f(c)$, Basic law **L9**.
(4) $f(c)$, [MP]: (3), (2).

A similar deduction could be used in order to justify the transition from $f(a)$ to $g(a)$, that is, the replacement of $f$ with $g$. In that case, the basic law **L9** would be formulated as follows:

$$\forall\mathfrak{F}\, \mathfrak{F}(a) \rightarrow g(a),$$

where $\Gamma(a)$ is the function in $\mathfrak{F}(a)$ and in $g(a)$.

This explanation does not directly proceed from *Begriffsschrift*. According to Frege's perspective, it would be possible to proceed straightaway from (1) to (4) and to consider this step as the result of an application of **L9**. The only difficulties that could arise are the particularities of each case of substitution, which should be resolved by the appropriate alphabetic replacements. Nevertheless, the fact that the substitutions of italic letters for italic letters can be reconstructed as a result of an application of [G] and [MP] and the use of **L9** shows the coherence of Frege's system.

There is one substitution of an italic letter for another letter worthy of remark: the substitution of a function letter for an argument letter, that is, for instance, of $F$ for $c$.[28]

The replacement of $c$ with $F$ is in many ways similar to other replacements of italic letters. However, an interpretation of *Begriffsschrift*'s concept-script as a second-order formal system does not explain it. The substitution of $F$ for $c$ entails taking $c$ as the argument of a proposition whose function must be capable of also taking function italic letters as arguments. Therefore, since the substitution of $F$ for $c$ has to preserve the assertibility of the proposition in which it is performed, in order to work it could require other replacements to be made in the proposition. We will illustrate this circumstance with an annotated example in Section 5.1.

4.3.2. The second case of substitution for italic letters consists in the replacement of a function italic letter with a complex expression. This means that the argument of a particular proposition is replaced with one of its instances. Concerning this case of substitution, it is convenient to distinguish between two sorts.

The first sort encompasses the substitution of a complex functional expression for a function italic letter.[29] These substitutions might entail dealing with several details. In particular, the instances of substitution of a function italic letter must inherit its arity and preserve the assertibility of the proposition in which the italic letter occurs. The fulfillment of these two circumstances, as we will see, might prove not at all to be a trivial matter.

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[28]In *Begriffsschrift*, Frege performs this particular substitution only in Chapter III: in the derivation of Pr. (77). We will discuss this derivation in detail in Section 5.1.

[29]In *Begriffsschrift*, applications of it can be found in the derivations of the following propositions: in §22, Pr. (65); §25, Prs. (70), (72) and (75); §27, Prs. (77) and (83); §28, Prs. (92), (97) and (98); §30, Prs. (109) and (110); §31, Prs. (116), (118), (120), (131) and (133).

A good example of the first sort of substitution for function italic letters is provided by the following. Consider basic law **L9**:

$$\text{Pr. (58)} \qquad \forall \mathfrak{a} f(\mathfrak{a}) \rightarrow f(c), \qquad \mathbf{L9}$$

where the functional expression $F(\mathit{\Gamma}) \rightarrow \forall \mathfrak{a}\, g(\mathit{\Gamma}, \mathfrak{a})$—which will be abbreviated as $\mathit{\Phi}(\mathit{\Gamma})$—is to be substituted for the letter $f$—in the form $f(\mathit{\Gamma})$—in **L9**.[30]

Firstly, both $f$ and $\mathit{\Phi}(\mathit{\Gamma})$ are unary. Secondly, $f$ occurs in **L9** inside the scope of a quantifier $\forall\mathfrak{a}$. This means that the substitution of $\mathit{\Phi}(\mathit{\Gamma})$ for $f(\mathit{\Gamma})$ would cause a clash in the quantifiers of $\mathit{\Phi}(\mathit{\Gamma})$ and **L9**. Thus, before this replacement is made, there must be an alphabetic change in **L9** that prevents this clash: for instance, $\mathfrak{a}$ could be replaced with $\mathfrak{d}$, from which results:

$$\forall \mathfrak{d} f(\mathfrak{d}) \rightarrow f(c). \qquad (8)$$

Finally, this substitution takes place in a derivation which contains symbols with a fixed interpretation: as we mentioned in Section 2.3, throughout Chapter III of *Begriffsschrift* $F$ is interpreted as a letter that stands for properties. Thus the places indicated by $\mathit{\Gamma}$ in $\mathit{\Phi}(\mathit{\Gamma})$, that is, in $F(\mathit{\Gamma}) \rightarrow \forall \mathfrak{a}\, g(\mathit{\Gamma}, \mathfrak{a})$, correspond to symbols that stand for objects. This circumstance is compatible with a possible reading of **L9**, according to which $c$ and $\mathfrak{a}$ are interpreted as letters that stand for objects.

Once $\mathfrak{a}$ has been replaced with $\mathfrak{d}$ in formula (8) and since $f$ and $\mathit{\Phi}(\mathit{\Gamma})$ have the same arity, no syntactic or semantic element concerning the assertibility of (8) is affected by the substitution of $\mathit{\Phi}(\mathit{\Gamma})$ for $f(\mathit{\Gamma})$. Therefore, this substitution is correct. If (8) is a valid formula, then the result of replacing $f$ with $\mathit{\Phi}(\mathit{\Gamma})$ in (8):

$$\forall \mathfrak{d}(F(\mathfrak{d}) \rightarrow \forall \mathfrak{a}\, g(\mathfrak{d}, \mathfrak{a})) \rightarrow (F(c) \rightarrow \forall \mathfrak{a}\, g(c, \mathfrak{a}))$$

is a valid formula as well.[31]

Frege performs this first sort of substitution for function letters without comments; neither does he make explicit the different steps we have developed nor detail the conditions that this substitution should observe. Even though this example of substitution can be applied to other similar replacements,

[30] Recall the use of $\mathit{\Gamma}$ as a device for easing the formulation of substitutions introduced in Footnote 26.

[31] This example is a simplified version of a substitution performed in the derivation of Pr. (70):

$$\text{Pr. (70)} \qquad \mathrm{Her}(F) \rightarrow (F(x) \rightarrow \forall \mathfrak{a}(f(x, \mathfrak{a}) \rightarrow F(\mathfrak{a}))),$$

where Pr. (68) is used as a premise:

$$\text{Pr. (68)} \qquad (\forall \mathfrak{a} f(\mathfrak{a}) \equiv b) \rightarrow (b \rightarrow f(c)).$$

In this derivation, Frege replaces $f(\mathit{\Gamma})$ with $F(\mathit{\Gamma}) \rightarrow \forall \mathfrak{a}(f(\mathit{\Gamma}, \mathfrak{a}) \rightarrow F(\mathfrak{a}))$ in Pr. (68). This substitution and the one we have exemplified are essentially the same. There is, however, an element worthy of consideration in the corresponding substitution for $f$ in Pr. (68): a binary function letter $f$ occurs in the instance of the letter $f$. In general, the fact that the letter to be replaced occurs in the expression by which it is to be replaced should be avoided. However, note that $f$ is a unary function letter in Pr. (68), while it is binary in the appropriate instance indicated in the derivation of Pr. (70). These two uses of $f$ cannot be conflated. Hence, in this particular derivation, the occurrence of $f$ in the functional expression with which it is to be replaced has no effect upon the development of the substitution.

to offer a global and rigorous definition of this sort of substitution would be tedious and tricky. The precise formulation of this definition may be complex, but the fact is that the practicalities of this kind of substitution pose no serious difficulties once the required alphabetic replacements have been made.

In the example we have just considered, the letter $f$ which has been replaced occurs in formula (8) together with letters such as $c$ and $\mathfrak{a}$; in this case, among other possible readings, $f$ can be interpreted as a predicate variable. Hence, from a contemporary perspective, the first sort of substitution for function italic letters can be seen as an application of a substitution rule in a second-order formal system. In contrast, some function letters in particular proofs of *Begriffsschrift* occur next to other function letters, as in $f(\mathfrak{F})$ or $g(F)$, and on those occasions they cannot be interpreted as a predicate variable. As a consequence, neither can these function letters be univocally identified with predicate variables nor may the substitutions in which they are involved be reconstructed in a second-order formal system with a substitution rule for predicate variables. This gives rise to a new sort of substitution, to which we will refer as the 'second sort of substitution for function italic letters'.

Chapter III of *Begriffsschrift* contains the most significant examples of the second sort of substitution for function italic letters. We discuss now one of them; furthermore, in Section 5.1 we will offer a detailed analysis of a derivation which includes a substitution of this sort. Consider again Proposition (52)—**L7**:

Pr. (52) $$(c \equiv d) \to (f(c) \to f(d)).$$ **L7**

Basic law **L7** is cited as a premise in several derivations of *Begriffsschrift*. In all of them Frege replaces $c$ and $d$ with formulas, that is, with expressions of assertible contents (see Footnote 13). In Section 2.3, we formulated in contemporary notation two different ways in which **L7** can be interpreted. However, on those occasions where Frege substitutes formulas for $c$ and $d$, the only compatible reading of **L7** is the following:

$$(\phi \leftrightarrow \psi) \to (\Phi(\phi) \to \Phi(\psi)). \tag{3}$$

Clearly, any substitution involving $f$ in **L7** in this context, namely $\phi$ in (3), could not be reproduced in a second-order calculus; it would not be the result of the application of a substitution rule for predicate variables.

Two prominent examples of the second sort of substitution for function letters are the substitutions of $\Gamma(y)$ for $f(\Gamma)$ and of $\Gamma$ for $f(\Gamma)$.[32] The former consists in the replacement of one formula with another; $\Gamma$ indicates the place corresponding to a function letter in the formula which is to be replaced. Note that the place indicated by $\Gamma$ must be occupied by a functional expression if the requirement of assertibility is to be met (see Footnote 26). The latter is a substitution of a propositional nature; whenever Frege proposes the replacement of $f(\Gamma)$ with $\Gamma$, the place indicated by $\Gamma$ corresponds to a

[32] The substitution of $\Gamma(y)$ for $f(\Gamma)$ takes place in the derivation of Pr. (93). The substitution of $\Gamma$ for $f(\Gamma)$ is more common in *Begriffsschrift*. It can be found in the following proofs: §22, Pr. (68); §25, Pr. (75); §28, Pr. (89); §29, Pr. (100); §30, Pr. (105).

formula.[33] Thus, both substitutions consist in the replacement of one formula with another.

Frege does not distinguish the two sorts of substitution for the function italic letters we have presented; on the contrary, he indicates the present sort in the same way as all other substitutions involving function italic letters. The author simply specifies which function letter is to be replaced and its instance; he uses, as we do, capital Greek letters in order to indicate the way in which every instance must fit in the resulting formula.

Nevertheless, from a contemporary perspective, these two sorts of substitution for function letters are of a different nature and do not admit a unitary treatment. The fact that Frege did not possess the resources that a second-order formal system provides is not properly significant here; all substitutions in *Begriffsschrift*, including those which belong to the sort we are considering, are intuitively valid.

4.4. In contemporary logic, the processes of substitution involve the formulation of several substitution rules and precise definitions of all possible substitutions. These definitions specify how the replacements can be performed, while the rules allow the deduction of an instance of substitution from a valid formula.

Strictly speaking, the concept-script of *Begriffsschrift* should contain definitions for the different types of substitutions we have taken into account. However, excluding a brief indication about alphabetic replacements, Frege does not provide any. We have stated that a substitution in the concept-script essentially consists in the replacement of a letter in all its occurrences in a formula with an appropriate instance. The author does not specify what an appropriate instance is, nor does he explain how in general to solve conflicts of quantification; these eventual conflicts are solved independently by means of alphabetic replacements without any comment.[34] The two aforementioned elements are indispensable in a rigorous definition of substitution in a formal system. However, since the language of the concept-script, taken in isolation, is not properly a formal language—as we put forward in Section 2.3—it is impossible to specify clearly and with detail what an appropriate instance is or to indicate how to perform substitutions in order to prevent any conflict of quantification. The plasticity of the letters of the concept-script, which is

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[33]The intuitive idea behind this substitution becomes clear with the help of an example. Consider this short derivation of the concept-script:

(1) $(c \equiv d) \to (f(c) \to f(d))$, **L7**

(2) $(c \equiv d) \to (c \to d)$, Substitution in (1) of $\Gamma$ for $f(\Gamma)$

This derivation would be expressed in contemporary logic in the following way: let $\phi$ and $\psi$ be formulas, $\Phi(\phi)$ a formula where $\phi$ occurs and $\Phi(\psi)$ the result of substituting $\psi$ for $\phi$ in $\Phi(\phi)$. The formula:

$$(\phi \leftrightarrow \psi) \to (\Phi(\phi) \to \Phi(\psi))$$

is a theorem for any formulas $\phi$, $\psi$ and $\Phi(\phi)$; since $\phi$ is a particular case of $\Phi(\phi)$, we can conclude that:

$$(\phi \leftrightarrow \psi) \to (\phi \to \psi)$$

is a theorem.

[34]There can be found in *Begriffsschrift* a couple of examples of alphabetic changes performed in order to avoid conflicts of quantification. See the derivations of Prs. (70), (116) and (118).

manifest in the wide variety of readings they have, prevents the formulation of a global definition of substitutions. This notwithstanding, we have explained why it is reasonable that Frege does not provide such a definition for the concept-script. Moreover, all substitutions in *Begriffsschrift* are intuitively correct.

Concerning substitution rules, many historical studies have discussed extensively whether Frege does use them in *Begriffsschrift* without making them explicit, and why he does not formulate them.[35] Under a preliminary analysis, the author should certainly include a substitution rule in the calculus of the concept-script. However, a historical explanation must provide a justification for this omission.

When the substitution rule is considered in contemporary commentaries to Frege's work, it is common to refer only to the substitution rule for predicate variables—which is closely related to the first sort of substitution for function italic letters. If the formulation of an inference rule for this case of substitution is deemed indispensable, then the corresponding inference rules for the remaining cases of substitution should be required as well. Nevertheless, contemporary historical studies only rebuke Frege for having omitted the substitution rule for predicate variables.

This attitude comes from the traditional reading of *Begriffsschrift*. According to this reading, the propositional fragment of the calculus demands a substitution rule, but it is avoided by formulating the corresponding basic laws as schemes and not as concrete formulas. It is understood that all propositional replacements taking place outside the propositional fragment of the concept-script can be performed in a similar way once all eventual conflicts of quantification are prevented. The substitution rule for argument letters, when they can be interpreted as individual variables, is deduced from a partial interpretation, in a first-order language, of basic law **L9**:

$$\forall x \phi(x) \rightarrow \phi(t),$$

where no occurrence of $x$ in $\phi$ is in the scope of a quantifier that bounds any variable occurring in the individual term $t$.

Therefore, from the perspective of the traditional interpretation of *Begriffsschrift*, only the formulation of what we will call the '*Substitution Rule*' is properly indispensable. This rule permits one to deduce a formula in which a functional expression—seen as a complex formula—has been substituted for a function italic letter—interpreted as a predicate variable—in a valid formula. This inference rule is fundamental for someone who defends the view that the formal system of *Begriffsschrift* is a second-order calculus, since it can be proven that it is equivalent to the Comprehension axiom.[36]

The second sort of substitution for function italic letters implies taking as a scheme the formula in which the substitution must be performed.

[35]See [43, p. 3], [9, p. 71], [6, pp. 334–337], [42, pp. 672–673], [35, pp. 185–186] and [28, p. 17].

[36]In [31] Henkin proves that the Substitution Rule is equivalent to a Comprehension axiom and thus its involved formulation can be avoided in the presentation of the standard second-order logic. This has become a common strategy in contemporary presentations of this particular logic. See also the sketches of proof made by Boolos [6, p. 337] and Russinoff [39, pp. 123–126].

The function letter to be substituted cannot be interpreted as a predicate variable (or it cannot be granted that this is its only possible interpretation). Therefore, the addition of the Substitution Rule to the concept-script calculus is insufficient, for some substitutions cannot be performed as an application of such a rule. Consequently, on the one hand, the reason for Frege's omission of an explicit formulation remains unexplained and, on the other, not all derivations of *Begriffsschrift* can be reconstructed. Moreover, if Frege's motivations not to make the Substitution Rule explicit are ignored, the rigor of his exposition, which is praised in almost every other context, is compromised. As a result, if the goal is providing a global and coherent reconstruction of *Begriffsschrift*'s substitutions, this strategy is plainly disputable.

## 5. Concept-script as second-order logic

The reconstruction of the axiomatic system of the concept-script we developed in the preceding sections can be verified in the application of this formal system to the deduction of theorems. In this context, it is possible to test the multiple interpretation of basic laws and inference rules as well as the particular character of substitutions. We already said that the language of the concept-script is not a formal language in the contemporary sense. In this section we analyze whether the concept-script can be seen as a second-order formal system once its language has been conveniently specified.

5.1. One of the most noteworthy deductions in *Begriffsschrift* is the proof of Proposition (77) [11, §27, p. 62; 174–175]. It is a key point in the discussion about the nature of substitutions in *Begriffsschrift* and the evaluation of the concept-script as a second-order formal system.[37] Once again, we will adapt Frege's notation as much as possible without altering the specificity of its formulas. Below we reproduce this deduction:

(1) $\forall\mathfrak{F}[\mathrm{Her}(\mathfrak{F}) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to \mathfrak{F}(\mathfrak{a})) \to \mathfrak{F}(y))] \equiv f^*(x,y)$,[38] Pr. (76).

(2) $(\forall\mathfrak{a} f(\mathfrak{a}) \equiv b) \to (b \to f(c))$, Pr. (68).

(3) $[\forall\mathfrak{F}[\mathrm{Her}(\mathfrak{F}) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to \mathfrak{F}(\mathfrak{a})) \to \mathfrak{F}(y))] \equiv f^*(x,y)] \to$
$[f^*(x,y) \to (\mathrm{Her}(F) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to F(\mathfrak{a})) \to F(y)))]$,
Substitution in (2):

| $\mathfrak{a}$ | $f(\Gamma)$ | $b$ | $c$ |
|---|---|---|---|
| $\mathfrak{F}$ | $\mathrm{Her}(\Gamma) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to \Gamma(\mathfrak{a})) \to \Gamma(y))$ | $f^*(x,y)$ | $F$ |

(4) $f^*(x,y) \to (\mathrm{Her}(F) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to F(\mathfrak{a})) \to F(y)))$, Pr. (77): [MP]: (3), (1).

There are some aspects in this deduction that should be clarified. All of them are related to the substitutions indicated in line (3) of the derivation. The first step in these substitutions is the replacement of $\mathfrak{a}$ with $\mathfrak{F}$. After a

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[37] See the accounts provided by Bynum [10, p. 286] and Heck [29].

[38] Recall the adoption of the abbreviation Her(F) introduced in Footnote 17.

Moreover, again following Boolos, we use $f^*(x,y)$ in order to replace the Fregean expression $\frac{\gamma}{\beta} f(x_\gamma, y_\beta)$, which states that $y$ follows $x$ in the series generated by the procedure $f$. This relation between $x$ and $y$ is commonly referred to as the strong ancestral relation, and it is defined by Frege in Pr. (76), which is used as a premise in this proof.

comparison between Propositions (76) and (68), it is clear that Frege intends Proposition (68) to be read in the following way:

$$(\forall\mathfrak{F} f(\mathfrak{F}) \equiv b) \to (b \to f(c)).$$

This reading reflects the application of Proposition (68) to a specific kind of German letters.[39] The replacement of $\mathfrak{a}$ with $\mathfrak{F}$ makes explicit that, in this particular case, all adequate instances of the German letter are functional expressions. This particular reading is distinguished in order to clarify which kind of expressions it is appropriate to consider in the context of this deduction. The substitution of $\mathfrak{F}$ for $\mathfrak{a}$, even if it seems odd to contemporary eyes, carries only a restriction on the instances of the German letter; it is not equivalent to a transition from a first-order quantification to a second-order one.

The second step indicated in line (3) of the derivation is the substitution of a complex functional expression for the letter $f$, that is, of $\mathrm{Her}(\Gamma) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to \Gamma(\mathfrak{a})) \to \Gamma(y))$ for $f(\Gamma)$. From Frege's perspective, given the fact that this substitution does not give rise to any conflict in the scope of quantifications, it is so natural that it goes without comment.[40] Nevertheless, this substitution is not the result of an application of the Substitution Rule in a second-order calculus: it is not an example of the first sort of substitution for function italic letters. The substitution we are considering now really consists in the replacement of $f(\mathfrak{F})$ and $f(c)$ with complex formulas. This means that since the interpretation of the unary letter $f$ must be uniform in the whole proof, neither $c$ nor $\mathfrak{F}$ can be interpreted as individual terms;

[39]This kind of reading can be applied to basic law **L9** as well; **L9** can be read in the following way:

$$\forall\mathfrak{F} f(\mathfrak{F}) \to f(c). \tag{9}$$

In *Begriffsschrift* we can find examples that support this claim. In the derivation of Pr. (93), Frege cites Pr. (60):

Pr. (60) $$\forall\mathfrak{a}(h(\mathfrak{a}) \to (g(\mathfrak{a}) \to f(\mathfrak{a}))) \to (g(b) \to (h(b) \to f(b))) \tag{10}$$

as a premise and uses it as follows:

$$\forall\mathfrak{F}(h(\mathfrak{F}) \to (g(\mathfrak{F}) \to f(\mathfrak{F}))) \to (g(b) \to (h(b) \to f(b))). \tag{11}$$

Frege does not need an independent proof of (11) in order to employ it in the derivation; formula (11) is simply an instance of Pr. (60) (of course, other substitutions are performed in (11) in order to replace the letters $f, g, h$ and $b$ with suitable instances).

Although Pr. (60) is not an instance of **L9** (it would only be so if its consequent were the logically equivalent formula $h(b) \to (g(b) \to f(b)))$, this detail is inessential and does not affect our claim here. Frege's use of Pr. (60) as (11) amounts to applying **L9** in the form (9). On the interpretation of **L9**, see Section 3.1.

[40]Note that the letter $f$ is unary in Pr. (68) and binary in its substitution instance, $\mathrm{Her}(\Gamma) \to (\forall\mathfrak{a}(f(x,\mathfrak{a}) \to \Gamma(\mathfrak{a})) \to \Gamma(y))$. This ambiguity appears exclusively after the use of Pr. (68), which is deduced in Chapter II of *Begriffsschrift* and contains a unary function letter $f$. This letter could be confused with the binary letter for procedures $f(\Gamma, \Delta)$, introduced in Chapter III. In order to avoid ambiguities, Frege could have reformulated Pr. (68) by replacing $f$ with $g$:

$$(\forall\mathfrak{a}\, g(\mathfrak{a}) \equiv b) \to (b \to g(c)).$$

However, in this case and in the context of Chapter III (as we have already noted in Footnote 31), the binary function letter $f$ has been explicitly attributed a particular use, which is far more specific than that of generic function letters. The different arities of the two different uses of letter $f$ also help to avoid any confusion.

this interpretation would be necessary if $f$ could be in turn interpreted as a second-order variable. In a contemporary formal system, Proposition (68) should be seen as the following scheme:

$$(\forall X\phi(X) \leftrightarrow \psi) \to (\psi \to \phi(Y)),$$

where the replacement of $f$ consists in the substitution of the proposed instance—that is, a functional expression that results in a formula when it is complemented with a function letter interpreted as a predicate variable—for $\phi(\Gamma)$.

The third step in the substitution table indicated in line (3) of the derivation proposes replacing $b$ with $f^*(x, y)$. It is clear that $b$ is propositionally interpreted in the whole deduction; its instance of substitution reflects this circumstance. There is no quantification that could affect this replacement and thus it is a trivial step.

The final step is the substitution of the letter $F$ for $c$. This provides additional evidence that function letters are, in particular deductions, appropriate instances of substitution of argument letters such as $c$. After all, $F$ and $c$ play the same role in the formulas standing in lines (3) and (2), respectively, of the derivation: in Frege's words, "[h]ere, in accordance with §10, $F(y)$, $F(\mathfrak{a})$ and $F(\alpha)$ are to be considered different functions of the argument $F$" [11, §27, p. 62; 175]. Therefore, a function letter occurring in a complex statement can be taken to be the argument, and this only has consequences in the determination of the kind of instances to which the statement can be applied. The substitution of $F$ for $c$ is legitimate as long as neither the structure nor the assertibility of the whole formula in line (3) change after the replacement. This is reinforced by the fact that this substitution does not involve the transition from a first-order to a second-order quantification; it is just a syntactic convention.

5.2. It is commonplace in contemporary historical studies of *Begriffsschrift* to defend its groundbreaking character and to emphasize that it should be considered a key work in the history of logic. The traditional interpretation of *Begriffsschrift* has maintained that once the basic laws and inference rules of the concept-script are conveniently reformulated, with the addition of an explicit Substitution Rule, one obtains a formal system by which the formal proofs contained in *Begriffsschrift* can be expressed in a second-order language.

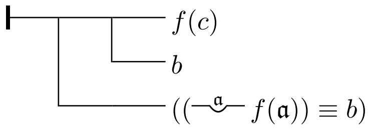
Our aim in this section is not to evaluate whether this thesis is technically acceptable, but to discuss its historical plausibility. Frege's purpose and practice are important to the clarification of this matter and they should not be minimized for the sake of an anachronistic correctness. The particularities of *Begriffsschrift*'s deductions reflect features of the concept-script and are completely coherent with the nature of this formal system. Defending the thesis that the concept-script can be faithfully reconstructed as a second-order formal system requires that one assume significant commitments regarding both its language and its axiomatic system.

In fact, as we have shown, there are substitutions in *Begriffsschrift* which cannot be performed as an application of a Substitution Rule in a second-order calculus. One example of these difficulties is the derivation of Proposition

(77). Bynum, in 'On an alleged Contradiction lurking in Frege's *Begriffsschrift*' [10], defends the case that the calculus of *Begriffsschrift* can be interpreted as a second-order calculus and, moreover, that its derivations can be reproduced in such a calculus by making only notational changes. According to Bynum, these notational changes clarify distinctions that, being developed in *Grundgesetze der Arithmetik* [16, 17] (hereinafter, *Grundgesetze*), were not available to Frege in 1879. Bynum thus offers an alternative formulation for the derivation of Proposition (77):

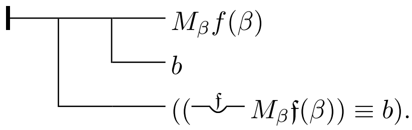
> "[W]hile proving formula (77), Frege cites the following principle (68):

Pr. (68)



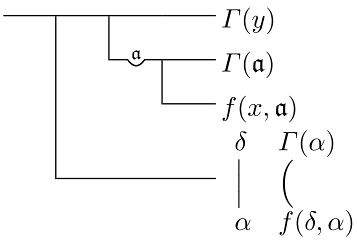
> Rather than this principle, he actually needs—but has not yet developed the machinery to express—an analogous second-order principle (call it 68′) involving quantification over functions. In the latter notation of the *Grundgesetze* (...) (ignoring the, for this purpose, irrelevant switch from '≡' to '=') it would look like this:

Pr. (68′)



(12)

> The appropriate substitution table (placed horizontally for convenience) would then run as follows:[41]

| $\mathfrak{f}$ | $M_\beta \varGamma(\beta)$ | $b$ | $f$ |
|---|---|---|---|
| $\mathfrak{F}$ |  | $\frac{\gamma}{\beta} f(x_\gamma, y_\beta)$ | $F$ |

> These substitutions in formula (68′), and the detachment of the definitionally true equivalence (76) in the *Begriffsschrift*, yields formula (77) with flawless correctness." [10, p. 286]

[41]For the sake of clarity, we include a reformulation in simplified notation of the substitution table Bynum proposes:

| $\mathfrak{f}$ | $M_\beta \varGamma(\beta)$ | $b$ | $f$ |
|---|---|---|---|
| $\mathfrak{F}$ | $\mathrm{Her}(\varGamma) \to (\forall \mathfrak{a}(f(x,\mathfrak{a}) \to \varGamma(\mathfrak{a})) \to \varGamma(y))$ | $f^*(x,y)$ | $F$ |

Concerning notation, see Footnotes 19 and 38. It is illustrative to compare this table with that which we have included in the exposition of the derivation of Pr. (77):

| $\mathfrak{a}$ | $f(\varGamma)$ | $b$ | $c$ |
|---|---|---|---|
| $\mathfrak{F}$ | $\mathrm{Her}(\varGamma) \to (\forall \mathfrak{a}(f(x,\mathfrak{a}) \to \varGamma(\mathfrak{a})) \to \varGamma(y))$ | $f^*(x,y)$ | $F$ |

Before we discuss Bynum's reformulation of this derivation, it is convenient to clarify a possible confusion concerning several notions related to quantification. In contemporary logic, the distinction between orders of quantification reflects a difference in the sort of quantified variables; a second-order quantification involves predicate variables. The distinction between function levels, absent in *Begriffsschrift* but completely and definitely stated by Frege in *Grundgesetze* [16, §§21–25, pp. 72–80], does not concern sorts of variables, but kinds of entities. In fact, functions in *Grundgesetze*—but not in *Begriffsschrift*—are a particular kind of entity which, unlike objects, have values for particular arguments. Functions are hierarchically ordered in levels according to the nature of their arguments. In Frege's words, "[f]unctions whose arguments are objects we now call *first-level functions*; in contrast those functions whose arguments are first-level functions will be called *second-level functions*." [16, §21, p. 37, Frege's emphasis]. In this way, a quantification over objects is in fact a function that has first-level functions as arguments and thus is a second-level function. Similarly, a quantification over first-level functions in *Grundgesetze*'s concept-script, which would correspond to a second-order quantification in contemporary logic, is a third-level function, that is, a function that has second-level functions as arguments.

Formula (12), proposed by Bynum:

$$(\forall \mathfrak{f} M_\beta(\mathfrak{f}(\beta)) \equiv b) \to (b \to M_\beta(f(\beta))), \tag{12}$$

belongs to the language of *Grundgesetze*'s concept-script and contains second-level function letters. The symbol $M_\beta(f(\beta))$ does not appear in *Begriffsschrift*: $M_\beta$ is a letter of the 1893–1903 concept-script that expresses generality over second-level functions (that is, functions in an absolute way) whose arguments are first-level functions [16, §25, p. 42]. In consequence, the presence of a symbol like $M_\beta(f(\beta))$ in (12) is not consistent with *Begriffsschrift* and, in fact, unnecessary.

Any acceptable interpretation of the derivation of Proposition (77) must be consistent with the conceptual scheme of *Begriffsschrift*. Functions in *Begriffsschrift* are simply that component of an expression that is taken at some point to be fixed, in contrast to the argument, that is, the component that is taken to be replaceable. Frege clearly states in *Begriffsschrift* that function and argument are components of given expressions [11, §9, p. 15; 126]. In fact, in the reconstruction of the derivation of Proposition (77) we have stressed how the distinction between function and argument changes when required in each step of the proof. This supports the thesis that no absolute sense can be attributed to the notion of function in *Begriffsschrift*.[42] Therefore, there can be no distinction between function levels in *Begriffsschrift*, neither implicit nor explicit.

As we have suggested, Frege does not need a modification of Proposition (68) such as (12). He states that a quantification in the 1879 concept-script involves only arguments; a quantification of a function letter is just a quantification of an argument—in the *Begriffsschrift* sense—that carries a restriction on the acceptable instances of the German letter. In this sense, only syntactic criteria distinguish a quantification of an argument letter from

[42] See a thorough analysis of this issue in [1, pp. 322–327].

that of a function letter. In order to explicitly indicate that the quantification of Proposition (68) involves functional expressions, it is enough to substitute $\mathfrak{F}$ for $\mathfrak{a}$ and $F$ for $c$ in (68):

$$(\forall \mathfrak{F} f(\mathfrak{F}) \equiv b) \to (b \to f(F)). \tag{13}$$

These replacements are exactly those which Frege adopts in the derivation of Proposition (77). Hence, it is indisputable that in *Begriffsschrift* argument letters can be read as function letters. We support the view that this possibility is coherent both with Frege's requirements of rigor and with his global conception of what the concept-script should be.

An adequate interpretation of formula (13) in a second-order language would be the following:

$$(\forall X \phi(X) \leftrightarrow \psi) \to (\psi \to \phi(Y)). \tag{14}$$

Note that since in Chapter III of *Begriffsschrift* $F$ is used as a letter that stands for properties, formula (13) is an application of Proposition (68) to the specific case in which the arguments are letters that stand for properties. Consequently, we can say that Frege already has (14)—or, at least, he has at his disposal the expressive resources in order to render its equivalent in concept-script—without modifying in the slightest the formalism of his formal system.

According to Bynum's account, the derivations of *Begriffsschrift* can be interpreted, save the details he points out, in a second-order formal system. From this perspective, Frege should have used formula (12) instead of Proposition (68) as a premise in the derivation of Proposition (77). In this context, the only significant difference between (12) and (68) is that the former explicitly includes quantification over function letters and notation that belongs to *Grundgesetze*'s concept-script, while the latter is an expression of the 1879 concept-script.[43]

Bynum seems to assume that the formulas of the concept-script can receive only one interpretation. According to what we indicated in Sections 2.3, 3.1 and several times since, this is completely unfounded. Moreover, Bynum does not provide any argument that supports the need to replace Proposition (68) with (12) in the derivation of Proposition (77); he does not explain Frege's procedure and aims in this proof but, on the contrary, he proposes an alternative by pointing out an alleged lack of rigor in Frege's version of the derivation. The appeal in this point to interpretative charity is unnecessary and condescending, because, on the one hand, Frege can obtain Proposition (77) without mentioning either function levels or formula (12)—or any alternative—and, on the other hand, his derivation is rigorous and correct. We support the view that the derivation of Proposition (77) in *Begriffsschrift* is clear and does not require any adaptation; it can be understood in the context of a global account of the concept-script. In particular, it is possible to understand it if (13) is interpreted in the suggested way.

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[43]Of course, the notational changes that Frege undertakes in *Grundgesetze* are not trivial. They reflect profound changes to the nature of the concept-script. A careful analysis of these changes goes far beyond the scope of this paper; we have restricted our analysis to highlight those elements that are relevant to the discussion.

Besides, Bynum's reconstruction of the derivation of Proposition (77) has to face greater difficulties. If, contrary what we have stated so far, it is maintained that the concept-script is a second-order formal system, the proof of Proposition (77) must follow the corresponding adaptation of the language and the axiomatic system. Moreover, this adaptation must be coherent with Frege's procedure in order to be historically sustainable. Bynum mentions that in order to correct Frege's derivation of Proposition (77), Proposition (68)—one of its premises—must be formulated in a second-order language and it thus should be the concept-script equivalent of (14). However, he does not take into account that the substitution of $\mathrm{Her}(\mathit{\Gamma}) \to (\forall \mathfrak{a}(f(x, \mathfrak{a}) \to \mathit{\Gamma}(\mathfrak{a})) \to \mathit{\Gamma}(y))$ for $f(\mathit{\Gamma})$ in Proposition (68), which is one of the steps involved in the proof of Proposition (77), is not reproducible using the Substitution Rule of a second-order calculus. In particular, Bynum does not notice that this particular substitution requires one to interpret Proposition (68) as (14). This is a fundamental difficulty for Bynum's reconstruction.

It is out of the question that an equivalent to Proposition (77) formulated in a second-order language can be deduced in a second-order axiomatic system by means of an alternative deduction than that proposed by Frege. We do not want to question this possibility but to argue that the Fregean proof is not and cannot be faithfully reconstructed as a derivation in a second-order calculus. Our discussion about Bynum's reformulation of this proof and, in particular, about his treatment of Proposition (68), shows that this adaptation entails changes that go far beyond a notational adjustment and the explicit formulation of the Substitution Rule.

The fact that, from a contemporary perspective, all these details have to be taken into account cannot be attributed to Frege's inaccuracies, because his deduction can be carried out, in the manner we have suggested, in plain coherence with his account. All substitutions set by the author are intuitively correct and admissible according to the formalism introduced in *Begriffsschrift*. In sum, Bynum neglects the nature of *Begriffsschrift*'s distinction between function and argument and, as a consequence, does not recognize the multiple readings that a single letter of the concept-script can receive.

## 6. Conclusion

Defending the claim that the deductive system of *Begriffsschrift* can be interpreted in second-order logic and, therefore, that all proofs contained in this work are reconstructible—except for minor details—in a second-order calculus entails contradicting key elements in Frege's exposition. From a general perspective, such a thesis demands, on the one hand, one to impose modifications on the language of the concept-script, based on the presence of a specific ontological structure, that is, the distinction between objects and properties; and, on the other, to reformulate the calculus.[44]

[44] We already considered the main aspects of the former in [1]; specifically, on the presence in Frege's 1879 work of a distinction in the language between objects and properties. We continue our analysis in this paper by focusing here on the syntactic aspects of the language and the reformulation of the calculus.

The components of the language of a second-order formal system reflect ontological categories such as that of object, property and relation. It is assumed that a property is not an object that can have properties of the same level and, as a consequence, symbols used to refer to objects and properties cannot receive an alternative interpretation. This categorization of the language is completely absent in *Begriffsschrift*: its letters can be read in different ways. Hence, a reformulation of the concept-script language in terms of a second-order formal language must restrict the multiple readings a single letter can have to just one and thus ignore its plasticity. This restriction cannot be done in general but rather should be adapted to the particular derivations in which each letter occurs. Therefore, depending on the context, an argument letter might be replaced with a propositional variable, an individual variable or a predicate variable; and a predicate variable or a meta-variable for formulas could be substituted for a function letter. This would force a reformulation of some of the basic laws.[45]

Moreover, in order to transform the language of the concept-script into a formal language in the contemporary sense, several syntactic distinctions need be introduced. Specifically, a sharp distinction between individual and predicate variables would be required, because without such a distinction it would be impossible to differentiate individual terms from formulas or to rigorously define syntactic notions such as substitution. However, this regimentation of the language of the concept-script is not only alien to *Begriffsschrift* but would also make the distinction between function and argument, which is *Begriffsschrift*'s cornerstone, lose its meaning and become useless.

Nevertheless, the most important modifications concern the axiomatic system. Specifically, the adaptation of the calculus of the concept-script to a second-order one requires:

(1) Splitting basic law **L9** into two parts, one for first-order variables and the other for second-order variables.
(2) Splitting the Generalization and the Confinement of the Quantifier inference rules into two parts, one for first-order variables and the other for second-order variables.
(3) Adding to the calculus a Substitution Rule for predicate variables, taking for granted that Frege does use it and, accordingly, he should have formulated it explicitly in *Begriffsschrift*.

In Sections 3.1 and 3.2, we argued that the changes included in the two first requirements—which are inseparably associated with the reformulation of the *Begriffsschrift* calculus—are unnecessary and, moreover, completely

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[45] In particular, this reformulation entails the elimination of the equality sign $\equiv$ and its replacement with a suitable symbol. Hence, basic laws **L7** and **L8** must be formulated as follows:

$$x = y \to (\phi(x) \to \phi(y)) \qquad \text{L7}'$$

$$x = x \qquad \text{L8}'$$

The alternative reading of **L7** we indicated in Section 2.3 cannot be expressed in a second-order language. However, recall that basic law **L7** and some of the theorems derived from it are used several times by Frege in a propositional way in *Begriffsschrift*, that is, according to the reading expressed in formula (2).

contrary to the spirit of the concept-script. Concerning the Substitution Rule, it is common to claim that Frege's omission of a Substitution Rule for predicate variables can be explained as epochal sloppiness. In Sections 4.3 and 4.4, we suggested that substitutions in *Begriffsschrift* can be explained as an instantiation of a universal quantification. In fact, we believe that from Frege's point of view the substitutions performed in this work are in no substantial way different from those common in mathematical practice. The substitutions performed in mathematics consist in the replacement of one expression with another of the same kind—just as *Begriffsschrift*'s substitutions—without there being rules that systematize these replacements. To Frege's eyes, the only significant added difficulty to the concept-script, namely, conflicts of quantification, is eventually solved by the author by appropriate alphabetic changes.

Besides, the lack of substitution rules should not be attributed to Frege's imprecision, since all steps made in *Begriffsschrift*'s substitutions are intuitively valid. The fact is that several substitutions can only be metalinguistically justified and hence are not the result of an application of a Substitution Rule in a second-order calculus. Therefore, the strategy of merely adding this rule to the concept-script necessarily implies the reformulation of the language of *Begriffsschrift* as a second-order language. Accordingly, the third requirement mentioned presupposes the first two. The particular nature of *Begriffsschrift*'s substitutions implies, on the one hand, that the calculus contained in this work is not a second-order calculus and, on the other, that to assume that the formal system of *Begriffsschrift* can be seen as such is historically both incorrect and inadmissible.

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