Frege's *Begriffsschrift* and logicism

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Abstract: I put forward a new interpretation of Frege's use of the formal system developed in *Begriffsschrift*, the concept-script. In contrast with the commonly-held view, I argue that this use suggests that he did not articulate a logicist programme in 1879. Two lines of argument support this claim. First, I show that between 1879 and 1882 Frege presented the concept-script of *Begriffsschrift* as a tool for arithmetic, and not as a logical theory from which to deduce arithmetical theorems. Second, I consider Frege's results in *Begriffsschrift* and conclude that they do not imply an endorsement of his later logicist programme.

Keywords: Frege, Logicism, Logistic, *Begriffsschrift*, concept-script

1 Introduction

It is commonly accepted that Gottlob Frege announced his logicist project in *Begriffsschrift* (1879a). Almost all historical studies agree that there Frege formulated his goal of showing that arithmetic is not an autonomous theory, but is based on logic alone[2]. The formal system developed in *Begriffsschrift*, the concept-script, is thus seen as the first step in the development of Frege's logicist programme.

In this regard, two elements are worth taking into consideration. First of all, without a characterisation of the logicist programme, Frege's endorsement of such a programme in *Begriffsschrift* cannot be adequately addressed. After all, several mathematicians and logicians contemporaneous with Frege agreed in one way or another that arithmetic was reducible to logic; the

[1]I am grateful to Calixto Badesa for his careful reading of the paper and helpful suggestions. Thanks to Aldo Filomeno, Juan Luis Gastaldi, Ansten Klev, Ladislav Kvasz and Vera Matarese for comments, and to Michael Pockley for linguistic advice.

The work on this paper was supported by the *Formal Epistemology – the Future Synthesis* grant, in the framework of the *Praemium Academicum* programme of the Czech Academy of Sciences.

[2]See, for instance, (van Heijenoort, 1967, pp. 1–2), (Dummett, 1991, p. 68), (Sluga, 1996, pp. 218–219), (Sullivan, 2004, p. 660) and (Blanchette, 2012, pp. 7–17).

inclusion of Frege in this trend would not mean much. Second, the claim that *Begriffsschrift* is the inaugural step in Frege's logicist project usually involves a noteworthy omission. Right after the publication of *Begriffsschrift*, Frege wrote several papers in which, among other things, he put forward a particular use of the concept-script that was briefly mentioned in *Begriffsschrift*: that this formal system could be applied to scientific disciplines – such as arithmetic or geometry – and improve both their expressive capabilities and their deductive rigour. The 1879–1882 papers were written in the context of a controversy between Frege and Ernst Schröder about their respective formal systems which emerged around the Leibnizian notions of *lingua characterica* and *calculus ratiocinator*[3]. Some historical studies identify Frege's attempt to create a *lingua characterica* with his assumption of the logicist thesis[4].

This paper is in two parts. In the first I shall characterise such an application of the concept-script: I shall consider the particularities of its language, the changes in interpretation of its symbols, and its deductions. In the second part, after a presentation of the basic elements of an articulated logicist programme, I shall claim, on the one hand, that the application of the concept-script is not compatible with such a programme and, on the other, that the results of *Begriffsschrift* do not show any endorsement of the logicist thesis.

2 Logistic: instrumental use of the concept-script

In the Preface to *Begriffsschrift* Frege associated the construction of the concept-script with a twofold goal. On the one hand, this formal system is a means to establish rigorous foundations for some propositions which are relevant in arithmetic and demonstrate that their proofs do not have to appeal to intuition (1879a, p. 104). On the other hand, Frege aimed at the construction of a formal structure fit to complement scientific languages, one that is capable of being used as an aid for the rigourisation of scientific proofs and the processes of concept formation (1879a, p. 106).

The result of the application of the concept-script to a given scientific discipline is a hybrid system which may be called *logistic*[5]. The use of the

[3] According to Frege, the fact that the application of the concept-script to a scientific discipline is capable of expressing content in a rigorous and unambiguous way is essential for considering the concept-script to be the basis of a realisation of a *lingua characterica*.

[4] See (Sluga, 1987, pp. 90–92), (Peckhaus, 2004, pp. 9–10) and (Korte, 2010, pp. 291–292).

[5] My use of the term 'logistic' is non-standard; it is introduced as a means to refer to Frege's instrumental use of logic (cfr. Church 1956, pp. 47–58). The notion of logistic has been traditionally opposed to the abstract use of logic (i.e., to symbolic logic). In his monograph *A*

concept-script as the basis of a logistic system by Frege is a central element in 'Anwendungen der Begriffsschrift' (1879b), 'Booles rechnende Logik und die Begriffsschrift' (1880) and 'Über den Zweck der Begriffsschrift' (1882).

3 Logistic: language

The language of the concept-script of *Begriffsschrift* is – to the contemporary eye – peculiar. First, in this work, Frege did not provide a definition of the notion of atomic formula. The most basic expression of the concept-script, '$f(a)$', can be interpreted in different ways. This is due to the generality expressed by the letters occurring in '$f(a)$'. In fact, the letter 'a' can be interpreted, depending on the context, as a sentential variable, as an individual variable or as a predicate variable[6]; the letter 'f' can also be subject to multiple readings that fit with those of 'a'. Therefore, it cannot be determined beforehand whether '$f(a)$' is an atomic formula or not.

Second, besides the letters, the language of the concept-script lacks non-logical symbols. In particular, there are no individual constants or predicate symbols in the language of the concept-script. Only the logical symbols – judgement and content strokes, connectives and the generality symbol – have a unique possible reading. Consequently, disregarding the propositional fragment of the concept-script, it is not possible to express any definite meaning by means of a concept-script formula; for instance, it is not possible to univocally express in a single formula of the concept-script that an object has some property[7].

In contrast, scientific discourses have non-logical symbols that refer to specific objects and relations. By means of them, it is possible to build atomic formulas. However, these discourses typically lack the formal resources

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Survey on Symbolic Logic (1918) Lewis presented a characterisation of logistic that fits with Frege's instrumental use of the concept-script:

> "[L]ogistic" is commonly used to denote symbolic logic together with the application of its methods to other symbolic procedures. Logistic may be defined as *the science which deals with types of order as such.* (...) Its subject matter is not confined to logic. (Lewis, 1918, p. 3)

[6]As illustrations of the possible interpretations of the letters 'a' and 'c', see the derivations of propositions (89), (92) and (77) in *Begriffsschrift*.

[7]Throughout Chapter III of *Begriffsschrift*, Frege employs the letter 'f' as a parameter for procedures. In this context, an expression such as '$f(x, y)$' would only be interpreted as an atomic formula: y is a result of the application of the procedure f to x. I shall consider this letter in Section 6.

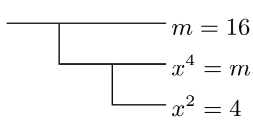
needed to express the logical relations that bind the atomic formulas together. They thus rely on natural language as a means to refer to complex notions or to define new concepts. Moreover, scientific discourses need the assistance of natural language in the construction of proofs (Frege, 1880, p. 13). In order to avoid the ambiguities and inaccuracies produced by the use of natural language, Frege proposed using the concept-script as the basis of what I term a system of logistic:

> What we have to do now, in order to produce a more adequate solution [than Boolean logic], is to supplement the signs of mathematics with a formal element, since it would be inappropriate to leave the signs we already have unused (...). Thus, the problem arises of devising signs for logical relations that are suitable for incorporation into the formula-language of mathematics, and in this way forming – at least for a certain domain – a complete concept-script. This is where my booklet [*Begriffsschrift*] comes in. (Frege, 1880, pp. 13–14)

Essentially, a system of logistic is the result of adding to the language of the concept-script the non-logical symbols of a scientific discipline (and thus, a means for building atomic formulas) and the construction of a calculus based upon the axiomatic system of the concept-script with the addition of some truths pertaining to the discipline in question. It is thus possible to understand why, in the exposition of the language of the concept-script of *Begriffsschrift*, Frege introduced neither a single non-logical symbol (i.e., individual constant, predicate or relation symbol) nor defined the notion of an atomic formula. This omission should not be seen as an epochal slip, since it is perfectly coherent if Frege's aim is observed. Departing from the atomic formulas of a given discipline, i.e., from those statements that contain no connectives or quantification, an expression in a system of logistic is built by using the logical symbols of the concept-script.

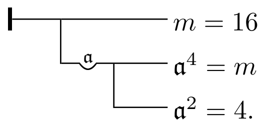
On several occasions Frege exemplified the application of the language of the concept-script to a particular discipline. In 'Booles rechnende Logik und die Begriffsschrift' (1880), he discussed a translation into logistic of an informal arithmetical statement:

> If every square root of 4 is a 4th root of m, then m must be 16.
>
> The expression



> does not correspond to the sentence, and is even false (...); for we may substitute numbers for x and m which falsify this content. (Frege, 1880, p. 18)

Frege wanted to highlight that an adequate logical analysis is an essential step when statements are to be formally expressed. The two different ways in which the concept-script can express generality render possible the appropriate symbolisation of the statement, "If every square root of 4 is a 4th root of m, then m must be 16". The resulting logistic expression is the following:

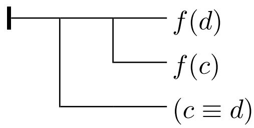


We can see in this example that Frege did not aim at symbolisation as it could be understood nowadays. In particular, he did not replace the non-logical symbols of the language of a given discipline with non-logical constants. In this sense, a system of logistic is not properly a formalised theory – such as Peano arithmetic. On the contrary, Frege just wanted to adequately render the formal complexity of the statements of a discipline by means of logical symbols while, at the same time, retaining the specific meaning these statements express. All atomic expressions, and the non-logical symbols occurring in them – symbols that denote, for instance, numbers and numerical operations – remain intact. This means that these non-logical symbols are not reinterpretable; since they do not acquire different meanings as do the individual constants and predicate symbols of first-order languages, the non-logical symbols in logistic should be considered canonical names.

4 Logistic: semantics

Scientific disciplines that are part of a system of logistic have a specific field of application which determines a domain of entities. The non-logical symbols of these disciplines denote objects in this domain, or relations and properties over objects in the domain. By means of these symbols and the presence of such a domain, the expressions of a system of logistic – unlike those of the isolated concept-script – acquire specific meanings and, in this sense, can be considered to be expressions of, as Frege put it, a *complete* concept-script.

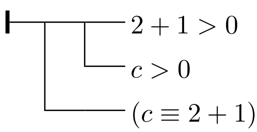
As stated above, a concept-script letter can be interpreted in different ways. Only when the formula in which a letter occurs is used in a specific context – paradigmatically in a derivation – can the relevant interpretation of the letter be determined. When concept-script letters occur in a logistic expression, their generality is restricted in two different ways. As an illustration, consider Proposition (52) of *Begriffsschrift*:



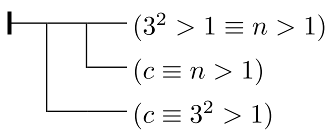
(52)

This proposition can be interpreted in at least two different ways: propositionally, or as a first-order logic formula.

Consider now the following instances of Proposition (52):



(A)



(B)

Both formulas could belong to the same system of logistic (in this case, the application of the concept-script to arithmetic). Note that, on the one hand, both (A) and (B) can only be interpreted in one way: the letter 'c' thus loses the multiple readings it can have in (52). In (A) 'c' is interpreted as an individual variable and, in contrast, it can only be propositionally interpreted in (B). On the other hand, the two incompatible interpretations of 'c' in (A) and (B) are also limited by the arithmetical language and the interpretation of arithmetical symbols. In particular, only numerical terms can take the place of the occurrences of 'c' in (A), whilst the appropriate instances of 'c' in (B) are arithmetical formulas.

Quantifiers in the applied concept-script are also adapted to the presence of a specific domain of entities and thus their meaning is restricted. All quantified letters still express generality, yet this generality is limited not only by the syntactic conditions imposed by the concept-script but also by the semantic restrictions laid down by the discipline.

This circumstance is apparent in 'Booles rechnende Logik und die Begriffsschrift' (1880). In this paper from Frege's *Nachlaß* the following example of definition, rendered as a logistic expression, can be found:

> The real function $\Phi(x)$ is continuous at $x = A$; that is, given any positive non-zero number $\mathfrak{n}$, *there is* a positive non-zero $\mathfrak{g}$ such that any number $\mathfrak{d}$ lying between $+\mathfrak{g}$ and $-\mathfrak{g}$ satisfies the inequality $-\mathfrak{n} \leqq \Phi(A + \mathfrak{d}) - \Phi(A) \leqq \mathfrak{n}$
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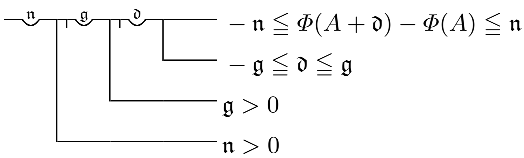
> 
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> I have assumed here that the signs $<, >, \leqq$ mark the expressions they stand between as real numbers. (Frege, 1880, p. 24)

This is an example of the application of the language of the concept-script to mathematical analysis. In this context the quantified letters '$\mathfrak{n}$', '$\mathfrak{g}$' and '$\mathfrak{d}$' are not individual variables that take values over an unrestricted domain, but letters that express generality exclusively over real numbers.

As a consequence, a statement in logistic – for instance, one which can be obtained from the previous example – is not logical, i.e., it is not a logical truth whose validity does not depend on a specific interpretation. A statement in logistic is intended to be true *only* in the discipline to which the concept-script is applied. In his examples, Frege did not only restrict the interpretation of the quantifiers but also avoided any reinterpretation of the non-logical symbols, which would be the usual practice nowadays. For instance, in the last example '$>$' is not taken to be a logical relation – applicable to any pair of objects – but a specific relation in mathematical analysis that is applicable only to real numbers.

5 Logistic: derivations

A system of logistic is not only the result of putting together the basic laws of the concept-script and a set of formulas that express facts about a discipline. Such a system would not be usable in derivations, since the basic laws of the concept-script do not contain any non-logical constant and thus their content is not connected with the domain of the discipline.

Actually, no proposition of the concept-script can directly participate in a derivation of a system of logistic. Only applications of the logical laws of the concept-script can be fruitfully used in a derivation that also involves formulas with a specific and unique reading. In order to render compatible the logical laws of the concept-script and the formulas of a system of logistic, some or all of the letters of those logical laws have to be replaced with the appropriate expressions containing those symbols found in the discipline. In other words, by means of substitutions, an expression of logistic is obtained as an application of a logical law of the concept-script. Through this process, a single interpretation of a logical law is fixed. This is why substitutions are essential in derivations in a system of logistic.

Therefore, only an application of the formulas of the concept-script is involved in derivations in logistic. This was shown to be the case by Frege in an example of a derivation he provided in 'Booles rechnende Logik und die Begriffsschrift' (1880, pp. 27–32); he first introduced the propositions of the concept-script that needed to be incorporated in the proof and he then indicated the substitutions that allowed the use of the appropriate instances of the logical laws in the derivation. In particular:

> In addition we need the formula (4) which is introduced as (96) on p. 71 of the *Begriffsschrift*. It means: if y follows x in the f-series, then every result of applying the procedure f to y follows x in the f-series[8]:
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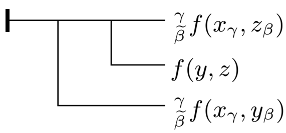
> 
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> (4)
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> (....) [W]e substitute $x + a = y$ for $f(x, y)$, 0 for x, $(n + b)$ for y and $(n + m)$ for z in (4), giving us (6)[9]:
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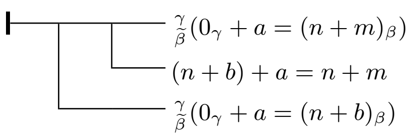
> 
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> (6)
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> (Frege, 1880, pp. 28–29)

Taken in isolation, there are no deductions with premises in the calculus of the concept-script: logical laws are obtained exclusively from basic laws and

[8]In this context, it can be of benefit to consider the successor function as an example of a procedure and the numerical order $<$ as an example of the series resulting from the application of the successor; accordingly, '$\frac{\gamma}{\beta}(n_\gamma + 1 = m_\beta)$' means that '$n < m$'.

[9]In this second formula, the procedure $x + a = y$ is defined by the operation $+a$: it relates a number with the result of adding a to it. Then the series associated to this procedure and starting with 0 corresponds to an ordering of the multiples of a.

other logical laws. However, a system of logistic does have a discipline's set of formulas which can be used as premises. In the above derivation in logistic, Frege set the goal of proving the theorem that "the sum of two multiples of a number is in its turn a multiple of that number" (1880, pp. 27–32). As a means to attaining this goal, he used two arithmetical laws as premises. Frege even distinguished these arithmetical laws from the "theorems of pure thought" he needed as logical laws in the proof – one of them, the formula (4) I have just considered. Accordingly, when a set of formulas of a discipline is available, then a deduction with premises can be considered in the calculus of the concept-script. As Frege clearly showed in this example, the premises used in a proof of logistic do not need to be axioms of the discipline. It is enough to isolate a set of formulas that are relevant in a given context, just as Frege did.

Once the syntactic structure of all formulas that take part in a derivation has been rendered uniform by means of substitutions, the inference rules of the concept-script can be used normally. After all, the distinction between function and argument, on which the inference rules of the concept-script of *Begriffsschrift* are based, is flexible enough to be applied to the expressions of any regimented language. Since the expressions of a system of logistic are constructed according to the syntactic rules of the concept-script – and, in particular, using its logical symbols – the application of inference rules is straightforward.

6 Concept-script and logicism

In this last section, after the consideration of how Frege intended to use the concept-script, I address the claim that he started his logicist programme in *Begriffsschrift*. The following passage of *Begriffsschrift*'s Preface is often cited as evidence for this claim:

> (...) I had first to test how far one could get in arithmetic by means of logical deductions alone, supported only by the laws of thought, which transcend all particulars. The procedure in this effort was this: I sought first to reduce the concept of ordering-in-a-sequence to the notion of *logical* ordering, in order to advance from here to the concept of number. (Frege, 1879a, p. 104)

The main goal in the creation of the concept-script in *Begriffsschrift* is usually considered to be the answer to the question of how far one could get in arithmetic by means of logical deductions alone. Therefore, from this perspective, the concept-script was created by Frege with the logicist programme in mind.

In light of Frege's exposition in 1879–1882 concerning the use of the concept-script, a reading of the last paragraph of the Preface to *Begriffsschrift* suggests a different diagnosis:

> Arithmetic, as I said at the beginning, was the starting point of the train of thought which led me to my "concept-script". I intend, therefore, to apply it to this science first, trying to analyse its concepts further and provide a deeper foundation for its theorems. For the present, I have presented in the third chapter some things which move in that direction. Further pursuit of the suggested course – the elucidation of the concepts of number, magnitude, and so forth – is to be the subject of further investigations which I shall produce immediately after this book. (Frege, 1879a, p. 107)

Frege's account fits with the explained instrumental use of the concept-script. In fact, Frege hinted in this passage at the two main elements that have been explained: on the one hand, the combination of the logical symbols and letters of the concept-script with the atomic statements of arithmetic to amend the shortcomings of the latter with regard to the process of concept formation; and, on the other hand, the reconstruction of arithmetical proofs by supplementing them with the formal resources of the concept-script calculus.

In order to assess whether this instrumental use of the concept-script is consistent with an endorsement of logicism, the basic elements of Frege's project of the reduction of arithmetic into logic should be expounded. A successful articulation of the logicist project demands the following[10]:

1. A specification of what is understood by logic, which includes a clarification of what constitutes a logical notion and a logical law;

[10] Frege never used the term 'logicism' (i.e., the German '*logizismus*'); logicism was first attributed to Frege as a foundational thesis of the reduction of mathematics into logic by Carnap (1931, p. 91). Interestingly, Carnap mentioned *Grundlagen der Arithmetik* (Frege, 1884) as the first work to advocate logicism. I am indebted to Marco Panza for this historical remark.

My account of Frege's logicist project does not mean to be exhaustive or definitive. The only goal of such an account is to serve as a minimal and clear model that enables the evaluation of Frege's position between 1879 and 1882. It is based on (Bays, 2000, pp. 415–416) and Badesa's unpublished material. Alternative approaches to the nature of Frege's logicism can be found in (Parsons, 1965), (Benacerraf, 1981), (Demopoulos & Clark, 2005), (Rayo, 2005) and (Kremer, 2006), among others. Concerning a historical reconstruction of the sources of Frege's logicism, see (Reck, 2013).

2. A logical calculus, composed of a determined set of basic laws and a limited and well-specified set of inference rules;

3. A justification that all basic concepts of arithmetic are logical notions, i.e., that they can be defined explicitly by means of logical notions[11];

4. A proof of all arithmetical laws in terms of the definitions obtained in (3) and the elements of the calculus specified in (2).

There is no trace of such an articulation of the logicist thesis in the papers written right after the publication of *Begriffsschrift* in 1879. In these papers Frege did not provide a single explicit definition of an arithmetical notion by means of logical notions; on the contrary, arithmetical concepts were defined using simpler arithmetical concepts. Therefore, arithmetic retains its basic notions and consequently its domain of specific objects, relations and operations. According to this, Frege maintained both the non-logical symbols of arithmetic and the restrictive interpretation of letters and quantifiers in complementing arithmetic with the concept-script, precisely with the aim of producing statements that refer to facts about numbers and the operations between them. Moreover, in the sole derivation of an arithmetical law that can be found in the papers written between 1879–1882 (the previously mentioned proof of the theorem that "the sum of two multiples of a number is in its turn a multiple of that number" (1880, pp. 27–32)) Frege used two arithmetical laws as premises in this proof. This is an implicit acknowledgement that the truth of the theorem to be proven is not founded on logic alone.

The results of Chapter III of *Begriffsschrift* raise another question regarding Frege's commitment to logicism in 1879. In this chapter he presented a relevant and particular example of the application of the concept-script: this formal system is used to obtain what Frege called "some propositions about sequences" (1879a, §23, p. 167). This is possible by providing symbols with a fixed, albeit abstract interpretation, such as, for instance, 'f' and 'F', which express generality over procedures and properties, respectively. Frege even introduced new letters, such as 'x', 'y', 'z' or 'm', which – unlike the regular letters of the concept-script taken in isolation – have a stable domain of interpretation: they express generality over objects. On this basis, Frege could define the notions of hereditary property, weak and strong ancestral, and single-valued (*eindeutig*) procedure, and derive some theorems that state basic properties of these notions, in particular, the principle of mathematical induction, i.e., Proposition (81) (1879a, §27, pp. 176–177).

[11] On the role played by explicit definitions in (3), see (Klev, 2017, pp. 342–344).

Frege had a calculus in *Begriffsschrift*, which allowed him to clearly specify what he understood by a logical method of proof and thus fulfil demand (2). However, the results of *Begriffsschrift* do not even show a partial commitment to logicism. Firstly, no justification that the basic notions employed in Chapter III are logical can be found in *Begriffsschrift*. Specifically, the notions of hereditary property or strong ancestral rely on the notion of procedure, which is introduced without any clarification: Frege merely translated into natural language a formula in which a binary function letter 'f' occurs[12]. Secondly, since Frege did not clarify what he understood by logic, he failed to justify the claim that the resulting theorems in Chapter III are logical laws in the sense specified in demand (4)[13].

All in all, the concept-script could eventually be tied to the development of the logicist thesis, but to serve as its vehicle was not its sole function. Not only did Frege explicitly intend to use this formal system in ways which are incompatible with a full assumption of logicism, but also only a few of the elements needed to fully articulate this thesis were present in *Begriffsschrift*, in which the concept-script was first introduced.

Pertaining to this, it is noteworthy that after the failure of Frege's logicist project, he kept using the concept-script, albeit in the more elaborate *Grundgesetze* version. According to Carnap's student notes (Reck & Awodey, 2004), Frege maintained the basic components of the concept-script – excluding the notion value range and Basic Law (V) – and used this formal system instrumentally, as a tool for the rigourisation of arithmetic.

Frege might have had intuitions concerning the logical nature of arithmetical truths in 1879–1882, and even before this period[14]. Given this, his

[12]The fact that Frege considered these notions to be within the realm of pure thought does not entail that he substantiated his position. In fact, I do not want to claim here that Frege ever successfully characterised the logical nature of those notions he took as basic. However, he explicitly addressed this matter from *Grundlagen* onwards – where the notions of object and concept are essential – and restricted their attributes to the ones he considered to be undeniably logical (see especially Frege, 1884, §27, p. 37, fn and Frege, 1884, §74, p. 87). See also the table of contents of the unfinished 'Logik' (1882–1891, p. 1).

[13]In *Begriffsschrift* Frege hinted at a basic idea of what he understood by logical laws, the "laws upon which all knowledge rests" and "transcend all particulars" (1879a, pp. 103–104); he pointed to their maximum generality (see also Frege, 1879a, p. 167). Frege first characterised in some detail the notion of logical law in 'Logik' (1882–1891, pp. 3–7) and further developed his position in other works and considered, at length, the justification of the laws of logic in *Grundgesetze der Arithmetik* (1893, pp. xiv–xix).

[14]In the list of theses Frege provided when he defended his dissertation "Über eine geometrische Darstellung der imaginären Gebilde in der Ebene" (1873), he included as the third thesis, "Number is not something originally given [*ursprünglich Gegebenes*], but can be defined".

reference to the reduction of the concept of number in the Preface to *Begriffsschrift* can be understood. However, Frege did not articulate these alleged intuitions either philosophically or formally. Besides, as a result of my analysis, I conclude that he could not have defended them as a programmatic goal without contradicting key features of his factual use of the concept-script.

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The complete list of theses can be found in (Kreiser, 2001, p. 123). I am indebted to Ansten Klev for this quotation.

In 'Booles rechnende Logik und die Begriffsschrift' (1880), Frege considered the possibility of using – as algebraic logicians did – the symbols of arithmetical operations to render logical relations. While examining this logical use of arithmetical symbols, Frege hinted at what could be seen as a logicist position:

> Anyone demanding the closest possible agreement between the relations of the signs and the relations of the things themselves will always feel it to be back to front when logic, whose concern is correct thinking and *which is also the foundation of arithmetic*, borrows its signs from arithmetic. To such a person it will seem more appropriate to develop for logic its own signs, derived from the nature of logic itself; we can then go on to use them throughout the other sciences wherever it is a question of preserving the formal validity of a chain of inference. (Frege, 1880, p. 12, author's emphasis)

Note that, besides this remark, Frege did not further develop his intuition regarding the nature of logic in 'Booles rechnende Logik und die Begriffsschrift' (1880). Actually, as I have showed, a significative part of this paper consists in a defense of the possibility of using the concept-script instrumentally.

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