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LINGUA CHARACTERICA AND *CALCULUS RATIOCINATOR*: THE LEIBNIZIAN BACKGROUND OF THE FREGE-SCHRÖDER POLEMIC

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Abstract. After the publication of *Begriffsschrift*, a conflict erupted between Frege and Schröder regarding their respective logical systems which emerged around the Leibnizian notions of *lingua characterica* and *calculus ratiocinator*. Both of them claimed their own logic to be a better realisation of Leibniz's ideal language and considered the rival system a mere *calculus ratiocinator*. Inspired by this polemic, van Heijenoort (1967b) distinguished two conceptions of logic—logic as language and logic as calculus—and presented them as opposing views, but did not explain Frege's and Schröder's conceptions of the fulfilment of Leibniz's scientific ideal.

In this paper I explain the reasons for Frege's and Schröder's mutual accusations of having created a mere *calculus ratiocinator*. On the one hand, Schröder's construction of the algebra of relatives fits with a project for the reduction of any mathematical concept to the notion of relative. From this stance I argue that he deemed the formal system of *Begriffsschrift* incapable of such a reduction. On the other hand, first I argue that Frege took Boolean logic to be an abstract logical theory inadequate for the rendering of specific content; then I claim that the language of *Begriffsschrift* did not constitute a complete *lingua characterica* by itself, more being seen by Frege as a tool that could be applied to scientific disciplines. Accordingly, I argue that Frege's project of constructing a *lingua characterica* was not tied to his later logicist programme.

§1. Introduction. It is common place in contemporary historical studies to distinguish two traditions in early mathematical logic.[1] As Zermelo put it in his lecture notes, 'Mathematische Logik':

> "The word "mathematical logic" can be used with two different meanings. On the one hand one can treat logic mathematically, as it was done for instance by Schröder in his Algebra of Logic; on the other hand, one can also investigate scientifically the logical components of mathematics." (Zermelo, 1908, p. 1)[2]

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[1] Unless otherwise stated, quotes are taken from the most recent English translation or edition listed in the bibliography. When an English translation is quoted, two page numbers—separated with a semicolon—are given: the first corresponds to the most recent original edition of the source and the second to the English translation. When no English translation is available or a published English translation is not used, quotes and page numbers are taken from the most recent edition of the source and translated by the author.

[2] Quote taken from (Mancosu, Zach, & Badesa, 2009, p. 320); English translation by Mancosu.

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The figure of Leibniz acted as an authority for early mathematical logicians. It was not uncommon to claim that one's own logic project was a fulfilment of Leibniz's notions of a *characteristica universalis* and a *calculus ratiocinator*. In fact, Jourdain, like Zermelo, differentiated two traditions in logic and tied each of them to one of Leibniz's components of the scientific ideal:

> "We can shortly but very accurately characterize the dual development of the theory of symbolic logic during the last sixty years as follows: The calculus ratiocinator aspect of symbolic logic was developed by Boole, De Morgan, Jevons, Venn, C. S. Peirce, Schröder, Mrs Ladd Franklin and others; the lingua characteristica aspect was developed by Frege, Peano and Russell." (Jourdain, 1914, p. viii)

In the first years of the 1880s two of the main proponents of these two traditions, Frege and Schröder, maintained a controversy concerning their respective formal systems. In response to the publication of Frege's *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (1879b),[3] Schröder wrote a long and mostly unfavourable review (1880). Frege replied in three papers: 'Booles rechnende Logik und die Begriffsschrift' (1881), 'Über den Zweck der Begriffsschrift' (1882b) and 'Booles logische Formelsprache und meine Begriffsschrift' (1882a), although only the second was accepted for publication.

One of the most prominent aspects of the polemic between Frege and Schröder is its background of the traditional notions of *lingua characterica* and *calculus ratiocinator*.[4] Both Frege (1881, p. 11; 10)[5] and Schröder (1880, p. 82; 10) claimed that their own formal system was a better realisation of Leibniz's ideal language and considered the rival system a mere *calculus ratiocinator*.

In a well-known paper (1967b), van Heijenoort drew a distinction, similar to Jourdain's, concerning conceptions of logic: he differentiated between logic as language and

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3 From now on, I shall use '*Begriffsschrift*' to refer to the book published by Frege in 1879 and 'concept-script' to refer to the formal system developed in it.

4 There are several ways in which Leibniz referred to his ideal language; the most renowned of them is '*characteristica universalis*'. However, in the nineteenth century—as exemplified by Frege and Schröder—it was common to use instead '*lingua characterica*' (or '*lingua characteristica*'). As stated by Patzig in the introduction to his edition of Frege's *Logische Untersuchungen* (Frege, 1966), the term '*lingua characterica*' did not come from Leibniz, but probably from Raspe's and Erdmann's editions of Leibniz's works (Leibniz, 1765 and Leibniz, 1840, respectively). In fact, Raspe entitled a 1678–1679 Leibniz's manuscript, 'Historia et Commendatio Lingua Charactericae Universalis' (Leibniz, 1765, pp. 533–540); this title has been preserved in later editions and translations. See (Kluge, 1977, p. 172). According to Patzig (Frege, 1966, p. 10, fn. 8), Frege took the term '*lingua characterica*' from the first part of the third volume of Trendelenburg's *Historische Beiträge zur Philosophie* (1867), the essay 'Über Leibnizens Entwurf einen allgemeinen Characteristik', which had been published as an independent work in (1856).

Haaparanta (2009b, p. 230) adds to this information that the word '*characterica*', instead of '*characteristica*', already appeared in Erdmann's edition of Leibniz's work as the title of some originally untitled works. Both Trendelenburg and Frege kept, in general, the term '*characterica*'.

5 See also Frege (1882c, p. 112; 88) and Frege (1882b, pp. 97–100; 91–93). In 'Über die Begriffsschrift des Herrn Peano und meine eigene' Frege still maintained his evaluation of Boolean logic: "[i]n Leibnizian terminology we can say: Boole's logic is a *calculus ratiocinator* but not a *lingua characterica*" (1897, p. 227; 242).

logic as calculus and presented them as opposing views. He ascribed to Frege the use of logic as language, whilst the use of logic as calculus was linked to the algebra of logic tradition. Essentially, van Heijenoort associated the construction of a *lingua characterica* with the development of a formal language that has the expressive devices—such as quantification—needed to analyse propositions (1967b, pp. 440–441). According to this conception, Boolean logic, seen as an early representative of the logic as calculus conception, could not provide a *lingua characterica*, whilst Frege had such a language from *Begriffsschrift* on. All in all, van Heijenoort limited himself to distinguishing two logical traditions, but did not provide either an adequate account of the conflict between Frege and Schröder or their diverging conceptions of logic.[6] Some recent attempts to characterise the polemic between Frege and Schröder, such as Peckhaus (2004a) and Korte (2010), do not seem to offer a completely successful account either. Specifically, textual evidence sullies with doubts Peckhaus' (2004a, pp. 9–10) and Korte's (2010, pp. 291–292) claim that Frege's aim to construe a successful *lingua characterica* was linked to his logicist project. In summary, I believe that none of these accounts provide an adequate reconstruction of the conflict between Frege and Schröder and, specifically, of their mutual accusation of having developed a *calculus ratiocinator*.

This paper focusses on the roots of the diverging meaning of mathematical logic suggested by Zermelo. However, I do not aim either to dispute van Heijenoort's general distinction between logic as language and logic as calculus, or to propose an alternative understanding of the two traditions in early mathematical logic. The purpose of this paper is specific: I shall provide an explanation of what Frege and Schröder meant when they accused each other of creating a mere *calculus ratiocinator* and of how they replied to these accusations. On the one hand, I shall explain what Frege and Schröder understood by *lingua characterica* and *calculus ratiocinator* and how they intended to realise these notions. On the other hand, I shall characterise Frege's and Schröder's accounts on the rival logic and offer a framework according to which these competing logical systems can be evaluated.

Schröder argued that a *lingua characterica* must have a set of basic notions from which all complex notions can be obtained. This idea is coherent with his conception of logic, which is focussed on the study of the algebraic properties of logical principles. Schröder's construction of the algebra of relatives fits with a project of the reduction of any mathematical concept to the notion of relative. I shall claim that this project, which was understood by Schröder as the realisation of a pasigraphy, explains his lack of interest in the formalisation of logic or in adopting a model theoretic point of view.[7] According to this characterisation, I shall argue that Schröder concluded that the concept-script of *Begriffsschrift* was a *calculus ratiocinator* because he saw no set

6 Hintikka further developed van Heijenoort's distinction and used it to characterise two traditions in twentieth-century logic. See Hintikka (1997b) and, especially, Hintikka (1997a) and Hintikka (1988). Sluga (1987) maintained and critically evaluated the understanding of the distinction between *lingua characterica* and *calculus ratiocinator* as their being two opposing conceptions of logic.

7 To my knowledge, Badesa was the first and has been almost unique in presenting together these two elements—(1) Schröder's pasigraphic project and (2) Schröder's indifference to metalogical issues—although he considers them in a broader context. See Badesa (2004, pp. 51–58). In a review of (Badesa, 2004), Jané departs from Badesa's analysis and establishes a causal relationship between (1) and (2) (2005, pp. 101–103).

of basic notions in *Begriffsschrift* and, consequently, he deemed the concept-script to be incapable, by itself, of deriving the complex notions of any scientific field.

Frege's early notions of a *lingua characterica* and *calculus ratiocinator* were not focussed on the reduction of all the concepts of a specific field, but rather on rigour, and expressive and inferential power. In characterising what Frege understood by expressive power I can explain why Frege considered Boolean logic to be an abstract logic, i.e., a logical system incapable either of expressing specific contents or of successfully capturing the processes of concept formation. I shall also claim that the language of *Begriffsschrift* concept-script was not seen by Frege as a complete *lingua characterica*, but as a tool that could be applied to the language of a scientific discipline and in this sense obtain a complete *lingua characterica* for a specific field. The study of the relationship between the concept-script and the discipline to which it is applied—paradigmatically, arithmetic—shall allow me to argue that Frege's project of the construction of the concept-script as the basis of a *lingua characterica* was not related to his logicist programme.

§2. Leibniz's *characteristica universalis*. Before I evaluate the opposition between Frege's and Schröder's accounts, I shall consider the Leibnizian origin and the nature of the notions of *characteristica universalis* and *calculus ratiocinator*. The purpose of this section is to describe Leibniz's project and to establish a common ground from which the notions of *lingua characterica* and *calculus ratiocinator* employed by Frege and Schröder can be better described.[8]

A *characteristica universalis* is, according to Leibniz, both a lingua franca to be used by all mankind and a tool for the rigorous and univocal expression of knowledge.[9] Leibniz stressed in 'Fundamenta calculi ratiocinatoris' that natural language is not suitable for this purpose; a *characteristica universalis* should be close to mathematical languages instead:

> "Ordinary languages, although considerably helpful for reasoning, are guilty of countless equivocations and cannot be used to perform the task of a calculus, namely, to allow the detection of errors of reasoning through the formation and construction of words themselves, as in the case of solecisms and barbarisms. Up to now, such admirable benefits are assured only by the symbols of Arithmetic and Algebra, where

[8] A historical reconstruction of the notions of *characteristica universalis* and *calculus ratiocinator* and of the adoption of these notions by logicians during the late nineteenth century is beyond the aim and scope of this paper. Concerning Leibniz's influence in early mathematical logic, see Lenzen (2004a, pp. 15–22) and Peckhaus (2014b).

[9] In the introduction to his edition of Leibniz's logical papers, Parkinson considered the distinction between *characteristica universalis* and *lingua universalis* (Leibniz, 1966, p. xvii). They are seen as essentially the same language; the only difference is that the *characteristica universalis* is a written language, whilst the *lingua universalis* is spoken.

For the sake of clarity, I simplify the ways of naming the universal language conceived by Leibniz. When I refer to this notion in the context of Leibniz's works, I use '*characteristica universalis*'. When I consider both Frege's and Schröder's accounts of this ideal language, I use '*lingua characterica*'. Finally, I uniformly use '*calculus ratiocinator*' to refer to the calculus associated with this language.

> all reasoning consists in the use of characters, and an error of the mind is identical with an error of calculation." (Leibniz, 1688–1689, p. 919; 182)

Leibniz conceived the creation of a *characteristica universalis* in different steps. First, the simplest concepts of every science must be identified. Second, the language must acquire a vocabulary made up by the symbols—or characters—that refer to the simple concepts. The final step consists in the formation of an appropriate system of representation of simple concepts and logical operations, which is necessary for obtaining complex concepts through the combination of simpler concepts.

In order to articulate the thoughts that make up human knowledge, it is necessary to build a symbolic structure which univocally depicts the organisation of the concepts from which this knowledge is built. The relationship between symbols and concepts has to be substantive, in the sense that the complexity of symbols must represent the structure of concepts. For instance, if the concept of human being is the result of the composition of the concepts of animal and rational, then the character for human being must show this composition. Then, since the characters can be related to numbers, the formation of complex concepts might operate as an application of arithmetical calculations: for instance, a new concept could be obtained from the multiplication of the numbers corresponding to two simpler concepts.

The *calculus ratiocinator* is a complementary tool for the language: it provides the means to rigorously verify whether a statement is derived from others in conformity with the properties of the operations that bind the concepts together. Hence, on the one hand, the *characteristica universalis* provides all the necessary means to formulate basic statements with the basic vocabulary and several operations between concepts; and on the other hand, the *calculus ratiocinator* regulates the reasoning based on those operations.[10]

As already noted, the elaboration of an alphabet of human thought is an essential task for constructing a *characteristica universalis*. Leibniz was concerned about the determination of the most basic concepts, and eventually concluded that a complete analysis of concepts, through which the primitive ones would be specified, is beyond the reach of human capabilities. However, he adopted the view that it is sufficient to establish those concepts which are required for the proofs of the truths of a specific subject matter.[11] Additionally, Leibniz suggested the possibility of working with a provisional set of basic concepts, or even to dissociate the calculus from specific content. In this sense, the *calculus ratiocinator* can be seen as a set of abstract rules of reasoning based on the establishment of relations between indeterminate concepts. According to Leibniz, without a *characteristica universalis*, the *calculus ratiocinator* becomes a sort of

[10] Note that the notion of *calculus ratiocinator* described here involves a systematisation of the notion of reasoning. In this sense, it should be distinguished from the notion of mathematical calculus, namely, from a system composed of mathematical objects and a set of operations that regiment the relations between the objects. A mathematical calculus can be axiomatised, but typically lacks a set of inference rules without which reasoning cannot be justified. By means of inference rules, every formal step performed in a proof can be made explicit and justified.

[11] See Leibniz (1966, pp. xxvii–xxviii). An example of a list of such purportedly primitive concepts can be found in 'Generales Inquisitiones de Analysi Notionum et Veritatum' (1686c, p. 744; 51).

abstract theory of classes.[12] Leibniz already considered this possibility in 'Fundamenta calculi ratiocinatoris':

> "Since this *ars characteristica*, whose idea I conceived, contains the True Organon of the General Science of everything that falls under human reasoning—when clothed with the uninterrupted demonstrations of a clear calculus—it will be necessary to expound our characteristic itself, i.e., the art of using signs by means of a certain kind of exact calculus, in the most general way. Since, however, it is not yet possible to establish how the signs should be formed, we will follow in the meanwhile the example of mathematicians, and use, for the signs which are to be formed in the future, letters of the alphabet or any other arbitrary symbols which progress may show to be the most adequate." (Leibniz, 1688–1689, p. 920; 182–183)

Given the particular nature of Leibniz's works, and the fact that a great portion of them was only available more than a hundred years after his death, his influence upon the history of logic has been limited. Only after Erdmann's edition of Leibniz's writings (Leibniz, 1840), and especially through Trendelenburg's essay (1856), did Leibniz begin to be known to German logicians. Frege and Schröder were eager to connect their own accounts of logic with a tradition in which Leibniz held a prominent place, but Leibniz's actual writings played a limited role in the development of their respective logical systems. One thus cannot claim that Leibniz's understanding of the notions of *characteristica universalis* and *calculus ratiocinator* shaped Frege's or Schröder's understanding of logic.[13] That said, the particular and occasionally incompatible readings of these two notions manifested by these two logicians reflect a substantive disagreement over their conceptions of logic. It is then relevant to study how Frege and Schröder understood the notions of *lingua characterica* and *calculus ratiocinator* in order to discuss the controversy regarding their rival logics.

§3. Schröder's logic as a pasigraphy. In this section I shall overview Schröder's conception of logic and address how he received and used the notions of *lingua characterica* and *calculus ratiocinator* to criticise the logic developed by Frege in *Begriffsschrift*.

3.1. Introduction. Schröder had already considered the ideal of a universal characteristic in his 1880 review of *Begriffsschrift*. His brief comments are close to Leibniz's account: Schröder described the creation of a universal characteristic as the process of obtaining all complex concepts from the least possible number of basic concepts—which he called 'categories' – using a set of rigorously specified operations

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[12] Leibniz worked on several versions of his calculus through many years. Some attempts to construct a calculus can be found in 'Ad specimen calculi universalis' (1686a) and 'Ad specimen calculi universalis addenda' (1686b), or in 'Generales Inquisitiones de Analysi Notionum et Veritatum' (1686c). See Rescher (1954), Lenzen (2004b) and Malink & Vasudevan (2016).

[13] On Leibniz's influence and, specifically, on the influence of the Leibnizian notions of *characteristica universalis* and *calculus ratiocinator*, in Frege's thought, see Patzig (1969), Kluge (1977) and Kluge (1980).

(1880, p. 81; 219). He carried on with this view and further developed it in 'On Pasigraphy'; in this paper Schröder summarised the task of a pasigraphy thus[14]:

> "The problem to be solved for any given branch of science amounts to: expressing *all* the notions which it comprises, adequately and in the concisest possible way, through a minimum of *primitive notions*, say "categories," by means of purely logical operations of general applicability, thus remaining the same for every branch of science and being subject to the laws of ordinary Logic, but which later will present themselves in the shape of a "calculus ratiocinator." For the categories and the operations of this "lingua characteristica" or "scriptura universalis" easy signs and simple symbols, such as letters, are to be employed, and—unlike the "words" of common language—they are to be used with absolute consistency (...)." (Schröder, 1899, p. 46)

In 1880, as we shall see, Boolean logic was not a fully developed formal system and presented significant shortcomings. Even though Schröder's perspective concerning a successful realisation of Leibniz's scientific ideal did not change substantially from 1880 to 1899, at the time of the publication of the review to *Begriffsschrift* he could not precisely determine the logical system he eventually aimed to construct; indeed, even though in *Der Operationskreis des Logikkalkuls* (hereinafter, *Operationskreis*) Schröder affirmed that Leibniz's ideal of a logical calculus had been realised in Boole's works (1877, p. iii), he recognised in his *Begriffsschrift* review that the scientific project envisioned by Leibniz had not yet been completed (1880, p. 81; 218–219).

Schröder published, in three volumes, his monumental treatise, *Vorlesungen über die Algebra der Logik* (1890; 1891; 1895; 1905) (the second part of the second volume (1905) was published posthumously). *Vorlesungen* brought to fruition the aim explicitly announced in 'On Pasigraphy' and are therefore the reference point concerning an evaluation of Schröder's conception of logic.

An overview of Schröder's logic is an unavoidable task if his conception of a *lingua characterica* is to be understood. To this end, my analysis distinguishes two different stages in the development of Schröder's logic. I consider Schröder's algebra of absolute terms to be the first stage. An early version of this logic was presented in *Operationskreis*, the main source of Schröder's position in 1880, when he wrote the *Begriffsschrift* review. An account of Schröder's algebra of absolute terms shall allow me to explain why Schröder took Frege's concept-script of *Begriffsschrift* as a mere *calculus ratiocinator*.

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[14] Traditionally, a pasigraphy is a language that, instead of representing sounds with symbols—like most natural languages—represents concepts. The first use of the word 'pasigraphy' (in French '*pasigraphie*') is attributed to Joseph de Maimieux, who in 1797 published *Pasigraphie ou Premiers élémens du nouvel art-science* (1797). Schröder understood a pasigraphy as "a scientific Language, entirely free from national peculiarities," by means of which "the foundation of exact and true philosophy" is established (1899, p. 45). In the first volume of *Vorlesungen über die Algebra der Logik* (hereinafter, *Vorlesungen*), he contrasted a pasigraphy, as a "*universal* language of things," with different natural languages (1890, p. 93); according to him, a pasigraphy should not be seen as a lingua franca of common use, but rather as a language of logical character (1890, p. 94, fn.).

Concerning Schröder's understanding of a pasigraphy, see Peckhaus (1991) and Peckhaus (2014a).

However, at this first stage Schröder did not have the means to construct a *lingua characterica* and hence could not fully address Frege's accusation of having created at most a *calculus ratiocinator*. Without considering a second stage, a substantive discussion of the polemic maintained between Frege and Schröder would be incomplete and one-sided. The second stage is Schröder's algebra of relative terms, the most complete and refined version of which can be found in the third volume of *Vorlesungen*. I take Schröder's account in 'On Pasigraphy' (1899) as representative of his later, more developed conception of logic. Only after a characterisation of Schröder's logic at this later stage shall I be able to explain his attempt at constructing a *lingua characterica* and thus complete my reconstruction of Schröder's project of the realisation of Leibniz's scientific ideal.

3.2. Schröder's algebra of absolute terms. Schröder's exposition in *Operationskreis* was heavily based on the work of Boole. In *An Investigation of the Laws of Thought* (1854) (hereinafter, *Laws of Thought*), Boole developed an algebra of the sum, the product and the difference, i.e., an algebraic theory that established the properties of these three operations by means of equations. The theory can be interpreted in different ways and, specifically, it can be applied both to the logic of classes (to which Boole referred as the 'logic of primary propositions') and to sentential logic (i.e., the logic of secondary propositions). The most relevant feature of these applications is that the logic of primary propositions is formally the same as the logic of secondary propositions.[15]

Schröder continued this equational algebraic system and presented it in *Operationskreis*. Following Boole, Schröder distinguished between judgements of the first class (*Urtheile der ersten Klasse*) and judgements of the second class (*Urtheile der zweiten Klasse*) (1877, p. 1). Again, a purely algebraic theory, which he called 'calculus of identity of domains of a manifold' (1880, p. 84; 221), can be applied to form, respectively, a calculus of classes or a sentential calculus.

Following his own account of what a universal characteristic should be, Schröder provided a set of basic operations or categories, by means of which the theory could be constructed. These essentially coincide with Boole's: equality ($=$), sum ($+$), product ($\cdot$), negation ($_1$) and the modules (0 and 1).[16] Division and subtraction were defined by means of the basic operations. The objects of these operations were symbolised by letters. All symbols corresponding to the primitive notions received a twofold interpretation depending on the application of the algebra: the calculus of classes or the sentential calculus. The equality symbol '$=$' was interpreted as equality between classes or as a symbol for logical equivalence; '$+$' as union or disjunction; '$\cdot$' as intersection or conjunction; '1' as the class of objects of thought under consideration (or Boole's universe of discourse Schröder, 1877, p. 7) or as "the time segment during which

15 Concerning Boole's logic, see Hailperin (1981), Hailperin (1986), Corcoran (2003), Hailperin (2004, pp. 349–361, 373–375) and (Brown, 2009). Although in his works Frege addressed Boole and Boolean logic, he did not engage in a personal discussion with the British logician, but with Schröder. After all, Schröder maintained and further developed the essential elements of Boole's logic. This is why in this paper I focus on Schröder as a representative of Boolean logic.

16 In later works Schröder modified his notation for negation. While in *Operationskreis* he rendered the negation of a as 'a_1' (1877, p. 10), in the first volume of *Vorlesungen* he used '$a_{\shortmid}$' (1890, p. 300). See Schröder (1890, p. 301).

the presuppositions of an investigation to be conducted are satisfied" (Schröder, 1880, p.87; 224); '0' as the empty class or an empty segment of time[17] ; and '$_1$' as complement or negation.[18] According to these interpretations, for instance, the formula '$a = a + b$' can be read as a judgement of the first class or as a judgement of the second class: in contemporary notation, as $a = a \cup b$ or as $\alpha \equiv \alpha \wedge \beta$ (where '$\equiv$' is the symbol of logical equivalence), respectively.

The theory presented in *Operationskreis* does not have a specific set of reasoning principles—i.e., logical axioms or inference rules—that can be distinguished from the algebraic theorems. Regarding this lack, although Schröder provided what seemed to be an axiomatic presentation in *Operationskreis*, it could hardly be considered a proper axiomatisation of Boolean logic.[19] Peirce advanced towards a true formal axiomatisation of this algebra in 'On the Algebra of Logic' (1880). An essential feature of Peirce's development is the introduction of the relation of subsumption ($-\!\!<$), by means of which equality can be defined.[20] This implies that equality was no longer regarded as the simplest logical relation. In fact, Peirce's incomplete axiomatisation was grounded on the $-\!\!<$ operation. In this sense, his adoption of subsumption as the basic logical operation was a significant advance towards the formulation of a formal theory and had noteworthy implications. On the one hand, the basic operations and the modules could be formally defined; on the other, it paved the way for a nonequational axiomatisation of the logic of secondary propositions. However, Peirce's account was hindered by the use of $-\!\!<$ both as a copula (i.e., as the connection between subject and predicate in a statement) and as illation (i.e., as the relation between the premises and conclusion in an argument) (1880, p. 170, fn. 5). The copula was interpreted as the relation of inclusion between classes, whilst the illation was interpreted as the relation of logical consequence and, crucially, also as the conditional. The identification of the copula with the illation shows that Peirce did not distinguish between the principles of reasoning and the algebraic laws; this affected his axiomatisation, since he presented as theorems several logical laws that, to the extent that they deal with notions that do not belong to the algebraic calculus, should be considered axioms.[21]

In *Vorlesungen* Schröder presented a complete formulation of the Boolean algebra of absolute terms, that is, the algebra that deals with simple objects or classes—and not

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17 The inclusion of the notion of time in the interpretation of the symbols in sentential calculus is Boole's solution in *Laws of Thought* which associates the logic of primary propositions with the logic of secondary propositions (1854, pp. 162–176). Schröder adopted the same interpretation in *Operationskreis* (1877, p. 1).

18 Hence, if U is the universe of discourse, the negation of a would be interpreted in the logic of classes as $a_1 = U - a$.

19 Peckhaus (2004b, pp. 587–588) arrives at a similar conclusion concerning the nonaxiomatic nature of the theory developed in *Operationskreis*.

20 Peirce introduced the relation of subsumption symbol '$-\!\!<$' in 'Description of a Notation for the Logic of Relatives, resulting from an Amplification of the Conceptions of Boole's Calculus of Logic' (1870). It was later, in (1880, p. 21, fn.), when he took this relation as more basic and simpler than equality.

Schröder had considered the relation of subsumption before *Operationskreis*, in *Lehrbuch der Arithmetik und Algebra* (1873, pp. 28–31). In fact, he even defined equality by means of subsumption. Even though this relation would be taken as a basic operation in *Vorlesungen*, Schröder decided not to include it in *Operationskreis*.

21 For a detailed analysis of Peirce's interpretation of the relation of subsumption, see Badesa (2004, pp. 14–15).

with relations[22] : in the first volume, he developed the calculus Peirce had outlined in 1880 and provided an axiomatisation of the calculus of classes; in the second volume, he put forward a sentential calculus by adding an axiom to the first volume's calculus of classes.[23]

In 1880 the language of the algebraic theory developed by Boole and refined by Schröder was, in a significant sense, incomplete. Its expressive power could not meet Schröder's aim, since it revealed important limitations. Schröder recognised one of these shortcomings in his review of *Begriffsschrift*:

> "There is a defect in Boole's theory, perceived by many (...), in the fact that particular judgements are only inadequately expressed in it (strictly speaking, not at all). The indeterminate factor *v*, which Boole uses, for example, in the first part of the logical calculus in the form $va = vb$ to express the sentence "Some *a*'s are *b*'s.", does not fulfil his purpose because, through the hypothesis $v = ab$, this equation always comes out an identity, even when no *a* is *b*. Now in the section concerning "generality", Frege correctly lays down stipulations that permit him to express such judgements precisely." (Schröder, 1880, p. 91; 229)

In *Laws of Thought* (1854, p. 61) Boole introduced the symbol '*v*', which was meant to represent an indefinite class and was used to translate the particle 'some'. As Schröder stressed, this symbol presented many problems, one considerable example being the inadequate analysis of particular categorical judgements.

As a consequence of what Schröder affirmed in the above quote, not *all* concepts could be expressed by means of a set of categories of Boolean algebra of absolute terms as of 1880. This explains Schröder's evaluation of this theory in his review of *Begriffsschrift*. Specifically, the Boolean algebra of absolute terms neither provided an adequate analysis of particular categorical statements, nor possessed the means to express relational concepts, nor possessed quantification. From 1880 all categorical judgements could be successfully analysed after the introduction of the notion of subsumption by Peirce. Schröder added quantification to the calculus of propositions

[22] Peirce distinguished between relative and absolute terms for the first time in (1870, pp. 364–365). Absolute terms express the properties of objects, whilst relative terms express relations between objects. According to Schröder, the notion of relative is equivalent to what in contemporary terms we call 'relation', namely, a class of ordered pairs (1895, p. 13). Concerning the notion of relative, see §3.4.

[23] As Huntington (1904, p. 291), Wiener (1913, pp. 26–27, 48) and Lewis (1918, pp. 119, 223–224) explicitly stated, Schröder's calculi of classes and propositions were determined by postulates, principles and definitions which globally constitute an axiomatic system. See Badesa (2004, pp. 20–21, 25–26), where the postulates of both calculi are listed and located in Schröder's works. See also Brady (2000, p. 144).

Wiener noted in his doctoral dissertation that some of Schröder's fundamental propositions of the calculus of classes "are given in the form of definitions with implicit existence-postulates" (1913, p. 27). However, this did not affect, in Wiener's view, the axiomatic character of the calculus. I am indebted to Calixto Badesa for these remarks.

Peckhaus (1996) offers an alternative view on the axiomatic character of Schröder's calculi of classes and propositions in *Vorlesungen* and, in particular, reconstructs Schröder's notion of axiom.

in the second volume of *Vorlesungen*.[24] Even so, a substantial number of concepts—those whose definition requires the use of relational concepts—could not be defined in the algebra of absolute terms. This means that many judgements could not be expressed by means of this algebra. In summary, the language of the algebra of absolute terms cannot be taken as a *lingua characterica*.

3.3. Concept-script from Schröder's perspective. Even though the Boolean algebra upon which Schröder's work was based in 1880 presented several expressive shortcomings, Schröder's critique of Frege's concept-script insisted upon its failure as a fulfilment of a *lingua characterica*. In fact, in the *Begriffsschrift* review Schröder deemed the concept-script to be a mere *calculus ratiocinator*:

> "[I]t must be said that Frege's title, *Concept-script*,[25] promises too much—more precisely, that the title does not correspond at all to the content [of the book]. Instead of leaning toward a universal characteristic, the present work (perhaps unknown to the author himself) definitely leans toward Leibniz's "*calculus ratiocinator*". In the latter direction, the present little book makes an advance which I should consider very creditable, if a large part of what it attempts had not already been accomplished by someone else, and indeed (as I shall prove) in a doubtlessly more adequate fashion." (Schröder, 1880, p. 82; 219–220)

The crucial point in Schröder's diagnosis is that he saw no vocabulary of basic notions in *Begriffsschrift*. This means that the language of this formal system, lacking symbols for the categories, is unable to define all complex concepts by means of a few. This is, in fact, quite accurate, since besides the logical symbols, the language of the concept-script only has letters, which cannot be considered categories in any sense.[26]

All in all, Schröder considered the concept-script to be a calculus of judgements (1880, p. 87; 224). In this sense he accused Frege of having created only a *calculus ratiocinator*. Even in this limited context he did not see that Frege had gone beyond Boolean logic. According to Schröder, Frege failed to provide a language rich enough to be able to define the complex notions of a discipline. Nevertheless, as I shall explain in §5.1, the concept-script, by means of its logical symbols—in particular, its theory of quantification—can successfully express all logical relations that help to build the derived notions of any scientific subject matter. Schröder did not appreciate that the concept-script had expressive resources that Boolean logic lacked. Moreover, Schröder failed to recognise the presence of a formal system in *Begriffsschrift*, that is, of a set of principles of reasoning (basic laws and inference rules) by means of which all formal steps in a proof could be rendered explicit. According to Schröder's 1877–1880 conception of a *calculus ratiocinator*, the principles of reasoning could be incorporated

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24 On the introduction of quantifiers into the algebra of logic, see Footnote 27.

25 For the sake of terminological homogeneity, I silently replace 'conceptual notation' with 'concept-script' in the quotes taken from Bynum's translations of Schröder (1880), Frege (1879b) and Frege (1882c).

26 To my knowledge, only Korte has successfully been able to explain what it means for Schröder that the concept-script is not a *lingua characterica*. See Korte (2010, p. 287). Later in that paper Korte suggests that the logical symbols of the concept-script language can be seen as its vocabulary of basic notions. I shall challenge this view in the following section.

in the calculus as transformation rules based on equalities. The concept-script, seen as a calculus of judgements, was for him a mathematical calculus (see Footnote 10). In this sense, the fact that the concept-script was adequate for being of assistance in proofs was unnoticed by Schröder.

This assessment of *Begriffsschrift* exposes the fact that Schröder undervalued its second chapter and, moreover, that he completely dismissed its third chapter. In chapter II of *Begriffsschrift*, not only the propositional apparatus of the concept-script is developed, but also its basic law regulating the use, in the calculus, of the notion of generality. In fact, Frege obtained in this second chapter the formal theorems required to conduct the proofs of the third. Schröder spurned Frege's unnecessarily complex notation and the generality of the results of chapter III (1880, pp. 92–93; 230–231), while he failed to recognise that Frege succeeded in justifying that the proof of some theorems that were instrumental in arithmetic did not rely on intuition. The lack of a careful study of the final chapters of *Begriffsschrift* could have caused Schröder's partial evaluation.

3.4. Schröder's algebra of relative terms. Peirce began the development of the logic of relatives as early as (1870). He made two essential steps for the algebraic development of logic and, in particular, for the progress of the algebra of relatives in his papers, 'The Logic of Relatives' (1883a) and 'On the Algebra of Logic: A Contribution to the Philosophy of Notation' (1885), namely the introduction of indices as individual variables and the interpretation of the generalised product $\prod$ and the generalised sum $\sum$ as quantifiers.[27]

[27] Euler (1755) first employed the '$\sum$' symbol as summation, while the use of '$\prod$' as product was introduced by Gauss (1812) (see Cajori, 1929, pp. 61, 78). Peirce associated these symbols to the quantifiers and, in doing so, he acknowledged the work of Mitchell—who first incorporated the distinction between 'some' and 'all' in the algebra of absolute terms in (1883). First, Peirce introduced the indices $i, j, k, \ldots$ as ranging over a specified universe of individuals. If x denotes an absolute term, then x_i is a new term—interpreted by Peirce as a coefficient—that takes the value 1 if 'i is x' is true and the value 0 if 'i is x' is false. Second, Peirce realised that the symbols for generalised sum and product can be used to symbolise the words 'all' and 'some' as products and sums of coefficients:

> "Here, in order to render the notation as iconical as possible we may use $\sum$ for *some*, suggesting a sum, and $\prod$ for *all*, suggesting a product. Thus $\sum_i x_i$ means that x is true of some of the individuals denoted by i or
>
> $$\textstyle\sum_i x_i = x_i + x_j + \text{etc.}$$
>
> In the same way, $\prod_i x_i$ means that x is true of all these individuals, or
>
> $$\textstyle\prod_i x_i = x_i x_j x_k, \text{etc.}$$
>
> If x is a simple relation, $\prod_i \prod_j x_{ij}$ means that every i is in this relation to every j, $\sum_i \prod_j x_{ij}$ that some one i is in this relation to every j (...)." (Peirce, 1885, p. 180)

Note that since in this context 0 and 1 are interpreted as the false and the true, respectively, sums and products of coefficients correspond to disjunctions and conjunctions, respectively, in the algebra of 0 and 1.

Schröder incorporated Peirce's use of the symbols '$\sum$' and '$\prod$' as generalised sums and products, respectively, into his calculus of propositions in the second volume of *Vorlesungen* (1891, pp. 26–48). In the third volume of *Vorlesungen* he extended the use of '$\sum$' and '$\prod$' to express sums and products of relatives (1895, p. 8).

Following Peirce, in the third volume of *Vorlesungen* Schröder expanded the language and the existing calculus of the algebra of absolute terms in order to build a system that provided an adequate logical analysis of relative terms. This expanded system's set of categories is the result of adding specific notions of the algebra of relative terms to the existing basic notions of the algebra of absolute terms. Each identical operation (that is, each operation that deals with absolute terms) has a relative counterpart. Accordingly, the primitive notions of this expanded set are the following:[28] identity ($=$) and relative identity (1'); product ($\cdot$), generalised product ($\prod$), and relative product (;); negation ($^{-}$) and conversion ($\breve{\ }$). The basic vocabulary, composed of the symbols for the categories, is enlarged in order to express derived notions in the language. These are sum ($+$), generalised sum ($\sum$) and relative sum ($\dagger$); subsumption ($\mathrel{\underline{\in}}$); and the modules (0, 1 and 0').[29]

While the relative operations ($\dagger$, ; and $\breve{\ }$) are only applied to relatives, the identical operations ($=$, $\cdot$, $\prod$, $+$, $\sum$, $^{-}$ and $\mathrel{\underline{\in}}$) can be interpreted either as operations between relatives or as operations between coefficients (see Footnote 27). A noteworthy example is that of the symbols '$\prod$' and '$\sum$', which can respectively be read as the universal and existential quantifiers when they are applied to coefficients—and not to relative terms. According to this reading, it is possible to quantify over individuals or relatives. This circumstance brings into consideration the two different domains over which $\prod$ and $\sum$ take values. The domain of individuals is denoted by 1^1 and is called the 'first-order domain' (*Denkbereich der ersten Ordnung*) (Schröder, 1895, p. 5). The second-order domain (*Denkbereich der zweiten Ordnung*) is the sum of all individual relatives and is denoted by 1^2 (Schröder, 1895, p. 10). Individual relatives are just classes whose unique element is an ordered pair of elements of the first-order domain.[30] Thus the relatives are the result of an identical sum of individual relatives.

Departing from these elements, Schröder obtained an algebraic theory, the theory of relatives, which contains as subtheories the calculus of classes and the sentential calculus that had been axiomatised in the first two volumes of *Vorlesungen*. The theory of relatives was never axiomatised by Schröder: he just offered in the third volume of *Vorlesungen* a set of postulates (1895, p. 17; pp. 22–35), which essentially consists in a collection of stipulations (*Festsetzungen*) that establish the meaning of the basic operations and informal clarifications that amend them.[31] In particular, Schröder provided two stipulations that define $\sum$ and $\prod$ as identical sums and products of relatives, respectively (1895, pp. 35–36). These stipulations relied on the interpretation

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28 These are the categories listed by Schröder in 'On Pasigraphy' (1899, pp. 47–49).

29 In set theoretical terms, 0 is the null relation, 1 is the universal relation, 0' is the diversity relation—which contains all ordered pairs composed of distinct individuals—and 1' is the identity relation. If U is the universe of discourse:

$$\begin{aligned} a\,;b &= \{\langle i,j\rangle : \text{ there is } k \in U \text{ such that } \langle i,k\rangle \in a \text{ and } \langle k,j\rangle \in b\} \\ \breve{a} &= \{\langle i,j\rangle : \langle j,i\rangle \in a\} \\ a \dagger b &= \{\langle i,j\rangle : \text{ for all } k \in U, \langle i,k\rangle \in a \text{ or } \langle k,j\rangle \in b\}. \end{aligned}$$

30 Schröder also introduced a third-order domain (1895, p. 14), which is the sum of individual ternary relatives. This third domain shall not be considered.

31 A detailed and thorough exposition of the postulates of Schröder's theory of relatives can be found in Badesa (2004, pp. 40–51). See also (Peckhaus (2004b, pp. 588–590).

of $\sum$ and $\prod$ as sums and products of coefficients, i.e., as quantifiers, which in turn were not defined by Schröder. He offered instead some explanations of the expressions 'some' and 'any' that quantified expressions obey (1895, pp. 36–37). Schröder acknowledged that these explanations were not real axioms, and thus that they did not belong properly to the theory of relatives—but then, the laws of quantification, which depended on these explanations, could not be proved in general in the theory of relatives.[32]

In fact, Schröder did not try to single out the formal apparatus of the theory of relatives. Even though this theory contained a logical language, Schröder never isolated it. Similarly, he did not distinguish the logical principles from the algebraic laws; he only provided a list of stipulations that regulate the meaning of the components of the language.[33] Crucially, Schröder did not establish a complete system of rules of inference by means of which the reasoning performed in the theory of relatives could be justified. In the second volume of *Vorlesungen*, he discussed the validity of some traditional modes of inference (1891, pp. 256–276), but he never developed a system of rules of inference by means of which the proofs could be fully regimented.[34] The theory of relatives was developed with the same kind of reasoning that is used in mathematical theories, where the reasoning principles are not explicit. Unsurprisingly, the lack of a formal apparatus for the theory of relatives was of no concern to Schröder.

Schröder pursued the development of the properties of the algebra of relatives, which is an abstract structure constituted of the aforementioned basic operations. Parallel to this interest was Schröder's belief that all mathematical objects could be viewed as relatives. As he put it in 'On Pasigraphy':

> "Almost everything may be viewed as, or considered under the aspect of, a (dual or) *binary relative*, and can be represented as such. Even statements submit to be looked at and treated as binary relatives.

32 By identifying quantified expressions with strings of sums and products and appealing to the properties of the identical sum and product, the laws of quantification could be proved only for a finite domain. Löwenheim saw this circumstance as a significant weakness. In his review of *Abriss der Algebra der Logik*, Löwenheim noted that "the rules of calculation for infinite sums of propositions" (that is, existential quantification over an infinite domain) had to be proved since Schröder "is unable to reach infinite sums and products without a fallacy" (1911, p. 72) (quote taken from Thiel, 1977, p. 244). On the derivation of the laws of quantification in Schröder's theory of relatives, see Badesa (2004, pp. 47–51).

33 Concerning the fact that Schröder did not isolate the formal or logical components of the theory of relatives, it is illustrative to compare his presentation of the theory with Tarski's (1941, pp. 74–76). In Tarski's formulation of the theory of relatives, the logical language (composed of variables for individuals and relatives, connectives and quantifiers) was carefully distinguished from the algebraic language (composed of the modules, identical and relative operations). Similarly, Tarski separated the logical axioms and inference rules—which were explicitly formulated—from the algebraic axioms. The latter "are intended to explain the meaning of the new constants" (Tarski, 1941, p. 75).

34 Although Huntington (1904, pp. 291, 297) carefully reconstructed Schröder's calculus of classes, he did not mention any inference rule. In fact, the proofs of some theorems that follow Huntington's presentation of the axioms of the calculus are carried out in the metalanguage. Similarly, Wiener formulated in this doctoral dissertation (1913, pp. 27–29, 47–48) all the axioms that constitute the calculi of classes and propositions—as presented in the first two volumes of *Vorlesungen*—and any reference to a complete system of reasoning composed of inference rules is lacking in his account.

> Classes, assemblages (Mengen, ensembles) or absolute terms may be thus presented.
>
> And since in ordinary as well as in scientific thinking the relative notions by far prevail over the absolute ones, which latter, over and above, are eventually comprised in and superseded by them, it is evident, that the Logic of the relative notions, Relatives, must form the indispensable base and underlie every successful attempt at Pasigraphy." (Schröder, 1899, p. 53)

This claim not only implies that all mathematical notions can be seen as relatives, but also that there is no need to deal with absolute terms at all. In fact, Schröder was convinced that each statement of the theory of relatives—containing indices, symbols for relatives, identical and relative operations—could be rephrased so as to obtain an equivalent expression in a restricted language which refers only to relatives and binary operations; the resulting expression would thus contain no indices or unitary predicates and all operations would be interpreted as operations between relatives. Schröder referred to this process as a compression or condensation (1895, pp. 550–551).[35] The resulting system—which I shall call the 'calculus of relatives' —is thus the product of eliminating in the theory of relatives all reference to individuals by means of the condensation of formulas.[36]

Schröder's pasigraphic project for the reduction of any mathematical theory, and in particular the theory of relatives, to the calculus of relatives can help one understand why he showed no interest in the axiomatisation of the theory of relatives or in formalising any mathematical theory. Schröder offered on many occasions examples of the generality and the expressive power of the calculus of relatives, by means of which several mathematical theories could be expressed. In the third volume of *Vorlesungen*, he translated Dedekind's theory of chains into the calculus of relatives (1895, pp. 346–387).[37] In this context, he declared:

> "The *final* goal of the work is to reach a strictly logical *definition* of the *relative* concept "*number of –*" [*Anzahl von –*] from which all propositions regarding this concept can be derived purely deductively." (Schröder, 1895, pp. 349–350; 299)

35 As Löwenheim stated in 'Über Möglichkeiten im Relativkalkül' (1915, pp. 448–449; 233–234), Korselt—a disciple of Schröder—had sent him a letter with a proof that there were formulas belonging to the theory of relatives that could not be condensed. Schröder's conviction was thus proven wrong. See also (Tarski, 1941, pp. 88–89).

36 My use of the terms 'calculus of relatives' and 'theory of relatives' is nonstandard, but has been adopted for the sake of clarity. I use these terms in the same sense as Badesa (2004, p. 53, fn. 27), which essentially corresponds to Tarski's (1941) use—although Tarski's characterisation of the calculus of relatives does not include quantification over relatives.

Concerning the process of condensation, see Goldfarb (1979, p. 354) and on the relationship between the theory of relatives and the calculus of relatives, see Mancosu *et al.* (2009, pp. 353–354). Also, on the connection between Schröder's pasigraphic project and the reduction of the theory of relatives to the calculus of relatives, see Badesa (2004, pp. 53–58).

37 Similar translations into the calculus of relatives can be found in Schröder (1898b) and Schröder (1898a).

Moreover, in 'On Pasigraphy' (1899, pp. 54–59) Schröder provided definitions of some of the most basic notions and facts in arithmetic expressed exclusively in the language of the calculus of relatives.

It is thus clear that Schröder aimed at a *reduction* in terms of the calculus of relatives, that is, at a definition of the basic notions of a mathematical theory in terms of relatives and operations between them, and a reformulation of the axioms and fundamental laws of the theory by means of the language of the calculus of relatives. This reduction should be distinguished from a *formalisation*. The latter requires a formal language and involves the disassociation of the nonlogical constants of the resulting expressions from the meaning of the primitive symbols of the formalised theory. Schröder never used the calculus of relatives for significant formalisations of mathematical theories.[38] Although Schröder had at his disposal all the technical elements required to isolate a formal language from the language of the calculus of relatives, he never attempted such a move. The possibility of attributing different interpretations to the nonlogical constants of a formal language was alien to his aim, but such attribution is instrumental in the formalisation of mathematical theories and the adoption of a model-theoretic point of view from which metalogical questions, such as the categoricity of a theory, can be addressed.[39]

38 A cautionary remark should be placed here. Even though the calculus of relatives has great expressive power, it is a system of enormous complexity. This circumstance could conceal the possibility of using the calculus of relatives for the formalisation of mathematical theories. In fact, the first true formalisation in the algebra of logic tradition was performed by Löwenheim in 'Über Möglichkeiten im Relativkalkül' (1915), and for that purpose he did use a restriction of the theory of relatives instead of the calculus of relatives. After the proof of the theorem named after him (1915, p. 450; 235), Löwenheim offered as an application the proof that all questions concerning the independence of the calculus of classes are decidable (if at all) in a denumerable domain (1915, p. 456; 240). To this end, he formalised the axioms of the calculus of classes. For instance, he took the axioms (1915, p. 457; 240):

$$a \mathrel{⋹} a, \tag{I}$$

$$(a \mathrel{⋹} b)(b \mathrel{⋹} c) \mathrel{⋹} (a \mathrel{⋹} c), \tag{II}$$

and formalised them as follows:

$$s_{aa} = 1, \tag{I}$$

$$s_{ab}s_{bc} \mathrel{⋹} s_{ac}, \tag{II}$$

where 's' is a binary relation symbol. Löwenheim stated that "since the axioms are to hold for arbitrary a, b, and c, we should still prefix $\prod_a$ (...) or $\prod_{a,b,c}$ to them" (1915, p. 457; 241).

Note that Löwenheim isolated the logical component of the language of the theory of relatives and carefully distinguished in his formalisation the use of '$⋹$' as a symbol for inclusion between classes (formalised as a relation symbol) and its use as conditional (which he left intact).

On the relation between Schröder's calculus of relatives and Löwenheim's formalisation of the axioms of the calculus of classes, see Badesa (2004, pp. 51–71) and (Jané, 2005, pp. 99–103).

39 In the first volume of *Vorlesungen* Schröder proved the independence of the distributive laws from the first seven axioms of the calculus of classes by proposing interpretations that satisfied the axioms and did not satisfy the distributive laws (1890, pp. 282–298, Anhänge

To the extent that the calculus of relatives is successful in reducing all mathematical concepts by means of the notions of relatives, this algebraic system can be seen as a pasigraphy and hence as a successful realisation of a Leibnizian *lingua characterica*. Furthermore, Schröder was convinced that the calculus of relatives was a *calculus ratiocinator*. This was based on two assumptions. He first assumed that the use of principles of reasoning essentially amounted to the application of identities that consist in rules of transformation of formulas assisted with propositional rules. These purported principles of reasoning were restricted to the calculus of relatives. This could not be seen by Schröder as a limitation, for his second assumption, as we have seen, was that that (almost) everything can be reduced to a relative.[40] Of course, this does not mean that Schröder defended this account when he wrote his review of *Begriffsschrift*; in 1880 he had not yet developed the calculus of relatives and, in fact, he did not even have a satisfactory axiomatisation of either the logic of classes or the sentential logic.

§4. Frege's conception of a *lingua characterica*. After the reconstruction of Schröder's critique of *Begriffsschrift*'s logic, I consider Frege's development of the notion of *lingua characterica*. This provides the necessary background for an explanation of Frege's view on Boolean algebra.

4.1. Evaluation of available languages. Frege's notions of an adequate *lingua characterica* and *calculus ratiocinator* are tied to the development of his formal system. In some of the papers written between 1880 and 1882 Frege used these concepts as tools for an evaluation of what a formal system should be and subsequently in his defence of the concept-script. In fact, in 'Booles rechnende Logik und die Begriffsschrift' (1881) he analysed in some detail the expressive capabilities a language must have to be considered a *lingua characterica*.

First, Frege shared Leibniz's and Schröder's rejection of natural language as a suitable tool for scientific means (1881, pp. 13–14; 12–13). In his view, this language allows for imprecisions both in the expression of conceptual relations and in the use of implicit presuppositions in proofs. Following Frege's example, two expressions which share a morphological structure, such as '*Berggipfel*' and '*Baumriese*' ('mountain top' and 'giant tree', respectively), can express different conceptual structures (1881, p. 13; 13). Conceptual expressions are not constructed in natural language in such a way that

4–6: 617–699). This could be considered a metalogical result. However, unlike Löwenheim's formalisation of the axioms of the calculus of classes (see Footnote 38), Schröder's proof amounted to the reinterpretation of the logical symbols occurring in the distributive laws and the axioms; it did not involve the use of a formal language. In this proof there is no reinterpretation of nonlogical constants. Accordingly, Schröder's proof of the independence of the distributive laws did not show the adoption of a model theoretic point of view.

On Schröder's proof, see Huntington (1904, pp. 291, 297–305), Peckhaus (1994), Thiel (1994), Badesa (2004, pp. 21–25). On the notion of the model-theoretic point of view, see Demopoulos (1994).

40 Schröder's assumptions on the calculus of relatives' capacity to justify reasoning should be qualified. The laws governing the operations between relatives cannot be used to verify the steps performed in inferences. The calculus of relatives did not include a complete system of inference rules and, therefore, it cannot be seen as a proper *calculus ratiocinator*. It should rather be seen as a mathematical calculus. On the distinction between *calculus ratiocinator* and mathematical calculus, see Footnote 10.

they reflect the structure of concepts. In fact, while the concept '*Baumriese*' consists in the intersection of the concepts 'to be a tree' and 'to be giant', the concept '*Berggipfel*' is only subordinated to the concept 'to be a top'.[41] In 'Über die wissenschaftliche Berechtigung einer Begriffsschrift' (1882c, p. 108; 84), Frege also alluded to the ambiguity of words; he mentioned the noun 'horse', which can be used to refer either to an individual, or to the species, or to a concept.

Second, Frege discussed the adequacy of mathematical languages as good candidates for use as a *lingua characterica*. Mathematical languages have a regimented syntax and a concise and univocal way of referring to basic concepts, which are essential features for the avoidance of ambiguities. However, these languages and, in particular, the language of arithmetic, lack those formal resources that are indispensable for the expression of the logical relations that help to define new concepts. In Frege's words:

> "The formula-languages of mathematics come much closer to this goal, indeed in part they arrive at it. But that of geometry is still completely undeveloped and that of arithmetic itself is inadequate for its own domain; for at precisely the most important points, when new concepts are to be introduced, new foundations laid, it has to abandon the field to verbal language, since it only forms numbers out of numbers and can only express those judgements which treat of the equality of numbers which have been generated in different ways." (Frege, 1881, p. 14; 13)

As Frege stated (1881, p. 30; 27), many complex concepts in arithmetic cannot be defined exclusively by means of arithmetical language, so they have to be incorporated in the theory with the assistance of natural language. The language of arithmetic's lack of expressive power is thus mended at the price of the imprecision and ambiguity of natural language. Moreover, the language of arithmetic has no way of formally conducting a proof and, again, must be complemented with natural language in drawing inferences.[42] The fact that arithmetical proofs are spelled out using natural

[41] I use in this context the same notation Frege used in 1880–1882 to render concepts or relations, even though it can induce errors given that the use of quotation marks conventionally serves to refer to the expression, and not to its meaning. In the works written after *Grundlagen der Arithmetik* (1884) (hereinafter, *Grundlagen*), Frege referred to concepts by means of italics.

[42] See Frege's remarks on this matter in 'Über die wissenschaftliche Berechtigung einer Begriffsschrift':

> "The arithmetic language of formulas is a concept-script since it directly expresses the facts without the intervention of speech. As such, it attains a brevity which allows it to accommodate the content of a simple judgement in one line. Such contents—here equations or inequalities—as they follow from one another are written under one another. If a third follows from two others, we separate the third from the first two with a horizontal stroke, which can be read "therefore" (...). Of course, this is by no means the only method of inference in arithmetic; but where the logical progression is different, it is generally necessary to express it in words. Thus, the arithmetic language of formulas lacks expressions for logical connections; and, therefore, it does not merit the name concept-script in the full sense." (Frege, 1882c, p. 112; 88)

See also Frege (1882a, p. 53; 47).

language means that there is no guarantee that their formal steps are fully explicit and, therefore, the correctness of the proof cannot be rigorously warranted.

To summarise, a language which realises Frege's notion of a *lingua characterica* has to fulfil both a material and a formal requirement. On the one hand, the language must render the basic concepts and the operations between them in a rigorous and unambiguous way—just like arithmetical language but, significantly, unlike contemporary formal languages, whose nonlogical symbols do not refer to any specific notion. On the other hand, obtaining complex notions requires the language to possess the resources necessary for the expression of statements in which those notions are defined. The calculus—seen as a *calculus ratiocinator*—is expected to rigorously justify how a judgement is obtained from other judgements in a proof while making explicit all principles of reasoning and implicit presuppositions. According to this account, the resulting language cannot be purely abstract; from Frege's point of view, the judgements of a *lingua characterica* must have a definite and specific meaning.

4.2. Boolean logic from Frege's perspective. Frege's particular understanding of how the language and the calculus of a logical system should be is a perspective which allows us to substantiate his criticism of the Boolean algebra of logic. The main focus of Frege's diagnosis is, in his view, the inability of the language of Boolean logic to express content.[43] In 1880–1882 the Boolean algebra of logic was a theory that studied the properties of the operations of the calculus of classes, which are common to those of the sentential calculus. The subject matter of the Boolean logic of classes consists of operations between concepts taken extensionally, that is, operations between classes that correspond to extensions of concepts. In this sense, algebraic logicians dealt with classes in an abstract sense, even though specific concepts were always mentioned in their examples. Boolean operations between concepts are not relative to any specific extension; they are meant to be applied generally. This is a crucial point, since it is the main reason for Frege's claim that Boolean logic is a mere *calculus ratiocinator*.

Therefore, according to Frege, Boolean logic should be considered 'abstract logic'; as he put it in 'Über den Zweck der Begriffsschrift':

> "When we view the Boolean formula language as a whole, we discover that it is a clothing of abstract logic in the dress of algebraic symbols. It is not suited for the rendering of a content, and that is also not its purpose. But this is exactly my intention." (Frege, 1882b, p. 100; 93)

The simple operations employed by Boolean logicians as categories cannot really serve to define all concepts in a scientific discipline; many concepts do not simply come from the intersection or the union of two previously available concepts.[44] Besides, the equational perspective again hinders to a great extent the expressive capabilities of the

43 See Frege (1881, p. 13; 12), Frege (1882b, pp. 97–98, 100; 90–91, 93) and Frege (1882a, p. 112; 88).

44 In a final summary of the unpublished (1881) Frege made this point explicit by affirming that "[the concept-script] is in a position to represent the formations of the concepts actually needed in science, in contrast to the relatively sterile multiplicative and additive combinations we find in Boole" (1881, p. 52; 46). Frege's critique of Boolean logicians regarding the process of concept formation is addressed to Kant—in very similar terms—in *Grundlagen* (1884, sec. 88, pp. 99–101). Frege even used the same spatial metaphor to substantiate his position in both texts. Concerning this metaphor, see (Wilson, 2010, pp. 392–393).

language. In this context, an adequate way of rendering quantification, as well as the propositional connectives, is essential. After the contributions of Peirce to the algebra of logic in (1883a) and (1885), these shortcomings were largely solved, at least as a matter of possibility.

Frege also considered the inadequacy of Boolean logic as a *calculus ratiocinator*. His idea of the rigourisation of a scientific proof not only involved the unambiguous and precise expression of the contents of premises and conclusion; he also referred, on the one hand, to the demand that all principles of reasoning and every formal step must be made explicit; and, on the other hand, to the fact that intuition should not play any part in any formal step (Frege, 1881, p. 36; 32). In fact, neither Boolean logic nor Schröder's algebra of absolute terms—as of 1882—possessed a proper formal system; these systems were not axiomatised and, in particular, they lacked a set of inference rules with which arithmetical reasoning could be fully regimented.[45] As we saw in §3.4, Schröder never produced a true *calculus ratiocinator*.

Frege's critique of the Boolean algebra of logic went beyond the limitations of its language and calculus. He also questioned its nature as a unified system of logic and, specifically, the relationship between the logic of classes and the sentential logic. In 'Über den Zweck der Begriffsschrift' he expressed himself in the following way:

> "Boole reduces *secondary propositions*—for example, hypothetical and disjunctive judgements—to *primary propositions* in a very artificial way. He interprets the judgement "if $x = 2$, then $x^2 = 4$" this way: the class of moments of time in which $x = 2$ is subordinate to the class of moments of time in which $x^2 = 4$. Thus, here again the matter amounts to the comparison of the extensions of concepts; only here these concepts are fixed more precisely as classes of moments of time in which a sentence is true. This conception has the disadvantage that time becomes involved where it should remain completely out of the matter." (Frege, 1882b, pp. 99–100; 93)

On Frege's critique of Boolean logic's inability to successfully render the process of concept formation, see Kremer (2006, pp. 174–177) and Heis (2013, p. 122). On a general account of Frege's view on concept formation, see Tappenden (1995).

45 In 'Booles rechnende Logik und die Begriffsschrift' Frege made explicit remarks on the completeness of Boolean calculus:

> "Schröder lays down the commutative and the associative laws of multiplication and addition as axioms in his '*Operationskreise des Logikkalkuls*', but doesn't derive from it for the case of more than three factors or summands that the order and grouping is arbitrary. But such proofs would be necessary, if you wished to prove in Boole's formal logic, as far as this is possible, the sentences derived by me, with an equally complete chain of inference. This wouldn't be afforded by 'mental multiplying out'. You also need the sentence that you may interchange two sides of an equation, and that equals may always be substituted for equals. Schröder does not include these among his thirteen axioms, although there is no justification for leaving them out, even if you regard them as self-evident truths of logic." (Frege, 1881, pp. 43–44; 38–39)

On the nonaxiomatic nature of the algebraic theory contained in *Operationskreis*, see §3.2.

Since the logic of classes and sentential logic are different applications of the same algebraic system, there should be a way of relating the statements of these two logics—primary and secondary propositions, respectively. Boole tried to relate these two kinds of propositions by appealing to what was—according to Frege—an unfortunate reduction of secondary to primary propositions. In contrast, Frege did not divide his logic into two parts that only diverge on the interpretation of the symbols, but instead provided a unique system of logic. The underlying reason for these divergent strategies is a different evaluation of the main object of logic: "[f]or in Aristotle, as in Boole, the logically primitive activity is the formation of concepts by abstraction, and judgement and inference enter in through an immediate or indirect comparison of concepts via their extensions" (Frege, 1881, p. 16; 15). As is well known, Frege rejected the idea that judgements come from the combination of concepts (1881, pp. 16–17; 14–16 and 1882b, pp. 100–101; 94) and defended, on the contrary, the view that concepts are obtained through the analysis of judgements. This novel perspective, which broke with tradition, freed logic from the constraints of operations between concepts—which had been deemed insufficient by Frege—and put the focus on the fruitfulness of the decomposition of statements, which were articulated by means of connectives and quantification.[46]

In summary, Frege's criticism of Boolean logic is thus based on four interrelated elements: the fact that its terms do not refer to the notions of specific disciplines, but to abstract classes; its inadequacy for expressing the logical relations that constitute the complex notions of a discipline; the lack of a formal apparatus needed for rigorously and precisely conducting a scientific proof; and the artificial reduction of secondary propositions to primary propositions. In contrast, the overcoming of these shortcomings is one of the leading motivations for the construction of the *Begriffsschrift* concept-script.

§5. Frege's concept-script as a *lingua characterica*. In this section I focus on Frege's attempt to realise Leibniz's scientific ideal. First, I characterise Frege's construction of a *lingua characterica* as the application of the concept-script to scientific disciplines. Second, I evaluate Frege's answer to Schröder's claim that the concept-script is only a *calculus ratiocinator*. Third, I defend the case that the way in which Frege used the concept-script in the papers written after the publication of *Begriffsschrift*, between 1879 and 1882, is independent of the logicist project.

5.1. Application of the concept-script. The logic of *Begriffsschrift*, the concept-script, is Frege's attempt to fulfil his ideal of both a *lingua characterica* and a *calculus ratiocinator*. As I shall explain in §5.3, one of Frege's main motivations in writing *Begriffsschrift* was, in fact, the desire to construct the formal structure necessary to

[46] Frege's perspective concerning the relation between judgements and concepts, according to which he "start[s] out from judgements, and not from concepts" (1881, p. 17; 16), has been called the 'principle of the priority of judgements over concepts'. Heis (2014) reconstructs the origin of the principle of the priority of judgements over concepts and offers a historical explanation of the influences that could have led Frege to formulate this principle. See also Sluga (1980, pp. 90–95), Schirn (1984), Sluga (1987, pp. 85–87), Haaparanta (2006) and Heis (2012, pp. 117–118).

complement scientific languages—one that is capable of being used for the rigourisation of arithmetic.

In many historical studies one point persistently ignored is that what Frege meant as a realisation of this ideal was the use of the concept-script as a tool for scientific disciplines.[47] This use is, nevertheless, so essential that it shaped the nature of the concept-script. As a formal system, the concept-script has a set of basic laws and rules of inference. However, in *Begriffsschrift* Frege neither introduced any nonlogical constant nor defined the notion of an atomic formula. Specifically, the language of the concept-script, as it was presented in *Begriffsschrift*, lacks individual constants and predicate symbols. In fact, the only nonlogical symbols of the language presented in *Begriffsschrift* are letters, which express generality.[48] In Chapter II of *Begriffsschrift* Frege exemplified the use of a language which, whilst being of use in expressing general logical truths, does not have the means to render any specific meaning. After all, no arithmetical theorem or definition can be expressed exclusively by means of letters (in contemporary terms, variables). The omission both of nonlogical constants and of the notion of atomic formula should not be seen as epochal slips, since they are perfectly consistent with Frege's aim. The concept-script was meant to be applied to a specific discipline and offer all the tools required for the rigorous definition of complex notions and the reconstruction of the proofs of the discipline. As the first step of such an application, the discipline provided the symbols needed to refer to its categories and derived notions. Accordingly, the language of the concept-script lacked a vocabulary of primitive concepts: it acquired the nonlogical symbols from the subject matter to which it was applied. Arithmetic was for Frege a paradigmatic discipline to which the concept-script could be applied. In his words:

> "Now I have attempted to supplement the formula language of arithmetic with symbols for the logical relations in order to produce—at first just for arithmetic—a concept-script of the kind I have presented as desirable. This does not rule out the application of my symbols to other fields. The logical relations occur everywhere, and the symbols for particular contents can be so chosen that they fit the framework of the concept-script." (Frege, 1882c, pp. 113–114; 89)

It is thus easy to understand why there is no definition of the notion of an atomic formula in *Begriffsschrift*. The lack of nonlogical constants (i.e., individual constants,

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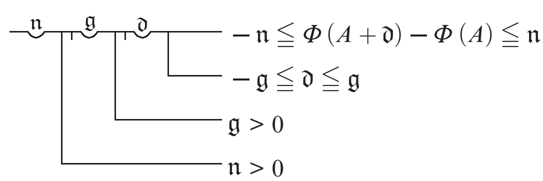
47 Dummett (1973, p. 630) hinted at this use. Nevertheless, the possibility of using the concept-script as a tool for the expression of scientific statements, which is central in Frege (1879a), Frege (1881), (Frege (1882c), Frege (1882b) and Frege (1882a), has not been the subject of a careful analysis.

48 Throughout chapter III of *Begriffsschrift*, Frege employed the binary 'f' as a schematic letter for procedures. A procedure is understood as a rule that once applied to an object of a specific range, returns one or more objects of the same rank. As a letter for procedures, the binary letter 'f' could be seen as a relation symbol. This particular use of 'f' is absent in chapter II, but at the same time is instrumental in the definitions provided in chapter III; in fact, it was introduced at the beginning of chapter III ((1879b), sec. 24, p. 57; 169). In terms of the present discussion, it is remarkable that Frege had to incorporate the binary 'f' in order to allow the language of the concept-script to define the notion of logical ordering and express the properties of sequences that constitute the content of chapter III. Note also that the notion of procedure remained undefined in *Begriffsschrift*.

predicate and relation symbols) was seen by Frege as a significant advantage, since it enabled the concept-script to be applied to a wide variety of possible subject matters and provide a *symbolisation* of their statements. This symbolisation was related to the formalisation which contemporary formal languages perform, particularly in the sense that they do not involve a reduction of a theory to a more general theory. However, Frege's symbolisation and a formalisation showed some significant differences. The language of the concept-script is not a formal language in the contemporary sense: Frege stated explicitly and insistently that when the concept-script is applied to arithmetic, he did indeed want to preserve the arithmetical symbols and their meaning.[49] In this sense, the symbols of the symbolised discipline are not replaced with nonlogical constants, but are kept in such a way that the resulting statements do express a content.

Frege devoted the second half of 'Booles rechnende Logik und die Begriffsschrift' (1881) to exemplify the ability of the concept-script to serve as a tool for a scientific discourse. He saw it as a distinctive advantage in comparison with the expressive capabilities of Boolean logic. For instance, he offered the following example:

> "The real function $\Phi(x)$ is continuous at $x = A$; that is, given any positive non-zero number $\mathfrak{n}$, *there is* a positive non-zero $\mathfrak{g}$ such that any number $\mathfrak{d}$ lying between $+\mathfrak{g}$ and $-\mathfrak{g}$ satisfies the inequality $-\mathfrak{n} \leqq \Phi(A+\mathfrak{d}) - \Phi(A) \leqq \mathfrak{n}$
>
> 
>
>
> I have assumed here that the signs $<$, $>$, $\leqq$ mark the expressions they stand between as real numbers." (Frege, 1881, pp. 26–27; 24)

In this example, atomic expressions of arithmetic remain intact. In particular, Frege did not replace arithmetical terms with nonlogical constants. The concept-script provided a rigorous means to express generality and also the logical relations that link these atomic expressions—in this case, conditional and negation.

Frege's aim was not only to symbolise—in the explained sense—arithmetical laws, but to provide all the formal resources required to reconstruct arithmetical proofs in purely formal terms. He conceived the application of the concept-script as the combination of two different elements: on the one hand, the incorporation of the syntactic resources of the concept-script to obtain a fully symbolised language and, on the other, the use of its calculus to render explicit all the formal steps in proofs. In 'Booles rechnende Logik und die Begriffsschrift' Frege offered a complete reconstruction of the proof of an arithmetical theorem—stating that the sum of two multiples of a number is in its turn a multiple of that number—using the formal

[49] Besides the quoted passage, see Frege (1881, pp. 14–15, 23–36, 51; 13–14, 21–32, 46) and Frege (1882b, pp. 100, 104; 93, 98).

resources of the concept-script (1881, pp. 30–36; 27–32).[50] To aim at a formal reconstruction of a proof means, in particular, that all its steps have to be made explicit and expressed by means of the logical formalism. Since some of these steps are purely logical—for instance, proceeding from a generalised statement to that particular case that is relevant in an inference—the concept-script has to provide a way to fill them. This is achieved by means of logical laws. Frege incorporated four logical laws in the above proof—two as premises and two as reasoning principles. These logical laws take the form of concept-script propositions, which cannot be directly applied to a nonlogical proof, since they only contain letters and logical symbols. Therefore, no proper proposition of the concept-script can be incorporated as a premise in the reconstruction of an arithmetical proof; one of its possible applications is used instead. These applications are obtained by means of substitutions: some of the letters occurring in a particular proposition of the concept-script are replaced with the relevant complex expressions, which are typically a combination of concept-script symbols and arithmetical symbols. In 'Über den Zweck der Begriffsschrift', while defending himself against Schröder's attack concerning the two-dimensional nature of concept-script symbolism, Frege explicitly mentioned how propositions of this formal system were incorporated into arithmetical proofs:

> "[M]y formula language indulges in the Japanese custom of writing vertically. Actually, this appears to be so, as long as one presents only the abstract logical forms. But if one imagines replacing the single letters with whole formulas, say arithmetical equations, he discovers that nothing unusual is presented here; for in every arithmetical derivation, one does not write the separate equations next to each other, but puts them one below the other for the sake of perspicuity." (Frege, 1882b, p. 104; 98)

To sum up, while the language of the concept-script can be used to formulate statements which express in a rigorous way the relationship that binds the notions of a scientific discipline together, its calculus makes it possible to reflect the inferential relations between the judgements of the discipline. The calculus provides the laws of reasoning—in the form of instances of concept-script propositions and inference rules—that enable one to rigorously conduct a proof of a theorem of a given discipline by making explicit all the formal steps. Since this calculus renders every step in a proof, it allows the elimination of natural language in inferences.

5.2. Concept-script and abstract logic. In 'Über den Zweck der Begriffsschrift', Frege claimed: "I did not wish to present an abstract logic in formulas, but to express a content through written symbols in a more precise and perspicuous way than is possible with words" (1882b, pp. 90–91). Frege's insistence on the need to express content can be understood in the light of the application of the concept-script and its

[50] In a sense, this proof can be seen as an analysis of the concept of being a multiple of a natural number. In fact, it shows the dependency between this concept and some more basic notions. Blanchette (2012, pp. 7–19) understands that this analysis is also a reduction, and relates it with Frege's logicist thesis. In the following section I argue that the application of the concept-script to arithmetic—for which this proof is one of the best examples Frege provided—does not show any endorsement of the logicist thesis.

use for the symbolisation of mathematical theories. This particular use demands the development of a formal system which contains all the logical principles and rules by means of which the inferences of a subject matter can be conducted. The concept-script is such a system, that is, a formal system with a set of basic laws and a set of inference rules that allow the proof of logical laws. In this sense, taken in isolation, the formal system of *Begriffsschrift* is an abstract logic:

> "In fact, I wanted to create not a mere *calculus ratiocinator*, but a *lingua characteristica* in the Leibnizian sense, and in so doing I indeed recognise the inferential calculus at least as a necessary part of a concept-script. If this was misjudged, it is perhaps because in the execution [of the concept-script] I stressed the abstract logical aspect too much." (Frege, 1882b, p. 91, author's translation)

Frege defended himself from Schröder's critique of the concept-script by claiming that, even though the establishment of the basic elements of the concept-script is an inescapable task, this is just one aspect of his whole project.

A development of logic that goes beyond an application to a scientific discipline—such as Frege's explained use of the concept-script—involves the discovery and formulation of the laws that establish the meaning of the logical notions. A paradigmatic example of this development would be the formulation of laws that characterise the relations between connectives and also quantifiers. Boole, Peirce and Schröder did formulate such laws.[51] In contrast, although he had all the expressive and logical means to do so, Frege did not systematically study the relations between connectives in *Begriffsschrift*.

Actually, one detail that shows Frege's indifference towards abstract logic is the fact that in *Begriffsschrift* he only obtained those logical laws that are strictly indispensable for the proofs in chapter III.[52] He did not analyse logical laws by themselves as if logic were its own subject matter.

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[51] See, for instance, Schröder (1895, pp. 76–101).

[52] In Frege's words:

> "I proved this proposition [Proposition (133)] through the definitions of the following in a series and the single-valuedness by applying my fundamental laws. At the same time I deduced the proposition [Proposition (98)] that in a series a member follows a second, if a third follows the former, then the third follows the second. Except for a few formulas that are introduced on account of the Aristotelian modes of inference, I only included that which appeared necessary for the aforementioned proof.
>
> These were the principles which guided me while setting up the primitive propositions as well as while selecting and deducing the others. For me it was completely trivial whether a formula seemed interesting or meaningless. That my propositions have enough content, if one can actually speak of content with regards to purely logical propositions, follows from the fact that they suffice." (Frege, 1881, p. 43, author's translation)

Frege alluded implicitly to Propositions (64), (65) and (66) of *Begriffsschrift*, which—as he said—were used to exemplify different Aristotelian syllogism modes. Concerning the lack of content of *Begriffsschrift* propositions, see Badesa & Bertran-San Millán (2017).

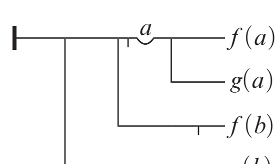
This circumstance is particularly patent in Frege's treatment of the Felapton and Fesapo syllogism modes in chapter II of *Begriffsschrift*. From a logical perspective, these two modes are distinguished because they correspond to different forms of reasoning; they belong to different syllogistic figures, as the order of the terms in the major premise shows:

FELAPTON	FESAPO
No M is P	No P is M
All M are S	All M are S
Not all S are P	Not all S are P

However, Frege applied these two modes in the following way: let b be an arbitrary individual with a certain property:

$$\begin{array}{l} b \text{ is not } f \\ \underline{b \text{ is } g} \\ \text{Not all } g \text{ are } f \end{array}$$

Strictly speaking, this argument cannot be taken as a Felapton or Fesapo, because its premises are singular. The significant aspect is the fact that the premises 'No M is P' and 'No P is M' are not distinguished, since they are applied as 'b is not f'. However, the distinction between the two major premises of Felapton and Fesapo is of logical interest: in contemporary notation, these two judgements are formalised as '$\neg\exists x(Mx \wedge Px)$' and '$\neg\exists x(Px \wedge Mx)$' respectively, and in virtue of the commutativity of the conjunction one can justifiably claim that they share a meaning. This logical property is of no interest to Frege's account; it is enough to have one single symbolisation for both premises and, thus, for both arguments:

> "We see how this judgement
>
> 
>
>
> (59)
>
> replaces one mode of inference, namely Felapton or Fesapo, which are not differentiated here since no subject is distinguished." (Frege, 1879b, sec. 22, p. 51; 163)

In sum, although it is clear that Frege needed to develop a formal system in *Begriffsschrift*, it would be wrong to understand that he only aimed at obtaining an abstract logic, that is, that he intended to construct a formal system in the contemporary sense.

As I explained in the previous section, the concept-script, unlike Boolean logic, was created as a general tool for the development of a discipline such as arithmetic. Its logical symbols and letters were expected to be combined with the nonlogical symbols of the language of this discipline, which made it possible to incorporate the propositions of *Begriffsschrift* as laws of reasoning in the proofs of the discipline. Before this application, the formulas of the concept-script, as Frege put it in 'Über den Zweck der Begriffsschrift', were "actually only empty schemata" (1882b, p. 103; 97) and its letters were expected to be replaced with relevant combinations of symbols in different contexts. In this way, the calculus—seen as an abstract formal apparatus—could be organically related to the *lingua characterica* that resulted from the combination of the linguistic resources of the concept-script and the atomic statements of a given discipline.

5.3. Concept-script as lingua characterica and logicism. Some historical studies claim that the core of Frege's aim of constructing a *lingua characterica* is the realisation of his logicist project by means of the 1879 concept-script.[53] In this sense, Frege's defence of the concept-script against Boolean logic would be closely connected with a vindication of a logicist programme announced in *Begriffsschrift*.

Frege explicitly addressed the claim that the concept-script was the basis of his fulfilment of the notion of *lingua characterica*. In order to argue for this claim, he showed that this formal system could be applied to scientific disciplines and, in particular, to arithmetic. However, in doing so, he did not consider the logicist thesis. Quite the contrary, the application of the concept-script—as Frege envisioned it—is incompatible in several ways with a full endorsement of the logicist programme.[54]

In the Preface of *Begriffsschrift* Frege associated the construction of the concept-script with a twofold goal. The main task of this formal system was to establish rigorous foundations for some propositions that are instrumental in arithmetic and prove that their justification does not need to appeal to intuition (Frege, 1879b, p. 8; 104). However, Frege also put forward a methodological goal: he aimed at the construction of a formal structure appropriate to complement scientific languages. This goal was associated with Leibniz's attempt to create a *characteristica universalis*:

> "Leibniz also recognized—perhaps overestimated—the advantages of an adequate method of notation {*Bezeichnungsweise*}. His idea of a universal characteristic, a *calculus philosophicus* or *ratiocinator*, was too ambitious for the effort to realize it to go beyond the mere preparatory steps (...). But even if this high aim cannot be attained in one try, we still need not give up hope for a slow, stepwise approximation (...). We can view the symbols of arithmetic, geometry, and chemistry as realizations of the Leibnizian idea in particular areas. The concept-script offered here adds a new domain to these; indeed, the one situated in the middle adjoining all the others. Thus, from this starting point, with the greatest expectations of success, we can begin to fill in the gaps in the existing formula languages, connect their hitherto separate domains to the province of a single formula language and extend it to fields which up to now have lacked such a language.

53 See Sluga (1987, pp. 90–92), Peckhaus (2004a, pp. 9–10) and Korte (2010, pp. 291–292).

54 In this section I would not like to attack the claim that Frege's endorsement of the logicist thesis started in *Begriffsschrift*. I only want to address the claim that Frege associated the construction of a *lingua characterica* with the development of the logicist thesis. Concerning Frege's endorsement of logicisim in *Begriffsschrift*, see (Bertran-San Millán, 2018).

> I am sure that my concept-script can be successfully applied wherever a special value must be placed upon the validity of proofs, as in laying the foundation of the differential and integral calculus." (Frege, 1879b, pp. xi–xii; 105–106)

Immediately after this passage Frege expressed his conviction that the concept-script could be successfully applied to geometry, pure kinematics, mechanics and physics as a whole (1879b, p. xii; 106). Considering these examples and particularly what has been discussed in §5.1, the use of the concept-script as the basis of a *lingua characterica* should be seen as an *application*, not as a *reduction*. Frege stressed on several occasions his intention to combine the logical symbols and the letters of the concept-script with the atomic statements of arithmetic. He also conceived the reconstruction of arithmetical proofs as the supplementation of arithmetical laws with those laws of reasoning required to render explicit every formal step.

Frege substantiated and exemplified the application of the concept-script, that is, its use as a basis of a *lingua characterica*, in the papers written after the publication of *Begriffsschrift*, between 1879 and 1882.[55] His exposition in these papers does not contribute to the deployment of the logicist thesis. Frege did not provide a single definition of an arithmetical notion exclusively by means of logical notions; on the contrary, he included explicit definitions of arithmetical concepts (1879a, pp. 990–93; 205–208 and 1881, pp. 23–29; 21–27), which only relied on simpler arithmetical concepts left undefined. Therefore, according to Frege's exposition, arithmetic retained its basic notions and consequently its domain of specific objects, relations and operations. In this regard, the generality expressed by the letters occurring in the formulation of arithmetical laws using the formal resources of the concept-script is always limited to the arithmetical domain.[56] Therefore, arithmetical laws were not treated as logical laws in any sense; their validity was considered only in an arithmetical context.

Moreover, in the sole proof of an arithmetical law that can be found in the papers written between 1879 and 1882—the proof of the theorem that "the sum of two multiples of a number is in its turn a multiple of that number" (1881, pp. 30–36; 27–32)—Frege used two arithmetical laws as premises:

> "The numbers whose multiples are to be considered are subject to no conditions other than that the following addition theorems:
>
> $$\vdash\!\!\!-\!\!-\!\!- \quad (n + b) + a = n + (b + a)$$
> $$\vdash\!\!\!-\!\!-\!\!- \quad n = n + 0$$
>
> hold for them (...). Of the theorems of pure logic we principally require that introduced as (84) on p. 65 of the *Begriffsschrift* (...). In addition we need the formula (4) which is introduced as (96) on p. 71 of the *Begriffsschrift*." (Frege, 1881, pp. 30–31; 27–28)

[55] See Frege (1879a), Frege (1882c), Frege (1882b), Frege (1882a) and, especially, Frege (1881).

[56] The restriction imposed upon the generality expressed by the letters occurring in arithmetical expressions can be verified in the example provided in §5.1. The papers Frege (1879a) and Frege (1881) contain other examples related to this.

Frege even distinguished the quoted arithmetical laws from the "theorems of pure thought" he needed as logical laws in the proof. In consequence, this inference cannot be taken as a proof that an arithmetical law is, in fact, logical; regarding this inference, Frege only set forth formal demands and stressed that "[p]recision and rigour are the prime aims of the concept-script" (1881, p. 36; 32).

All in all, I am not denying here that the concept-script could be tied to the development of the logicist thesis; I argue against the claim that to serve as the vehicle of this thesis was its sole and constitutive function. Disregarding the intuitions that Frege might have had concerning the logical nature of arithmetical truths,[57] Frege's account of the application of his concept-script, which was essential in the controversy he maintained with Schröder and in his articulation of the notion of *lingua characterica*, was independent of the logicist programme.

§6. Concluding remarks. Frege's conception of a *lingua characterica* was quite different from Schröder's; this circumstance hindered a substantive discussion between them. Their divergent understanding of this notion reflects two different conceptions of logic and of its function. In this paper, an analysis of Frege's and Schröder's conceptions of logic has provided a background adequate to explain the fact that they mutually accused each other of producing a mere *calculus ratiocinator*.

In his attempt to build a pasigraphy, Schröder intended to develop a language that could express every mathematical notion by means of a set of primitive concepts and operations. In his mature works he identified the notion of relative as the most basic and was convinced that, with the help of some operations, every mathematical concept could be defined as a relative. Schröder's pasigraphic endeavour was thus associated with the reduction of mathematical theories to the calculus of relatives. I argued that Schröder's claim that Frege's *Begriffsschrift* did not contain a *lingua characterica* was based on the lack of primitive notions in the concept-script. This claim was in fact fair, since, besides the logical symbols, the language of the concept-script did not have symbols for any basic notion. Schröder thus concluded that the concept-script was only an abstract structure—specifically, a calculus of judgements—incapable of rendering the complex notions of any science.

[57] In (1881) Frege considered the possibility of using—as algebraic logicians did—the symbols of arithmetical operations to render logical relations. He ruled out this possibility, for it would entail the symbols of arithmetical operations being able to have two different meanings in a single formula. While examining this logical use of arithmetical symbols, Frege hinted at what could be seen as a logicist position:

> "Anyone demanding the closest possible agreement between the relations of the signs and the relations of the things themselves will always feel it to be back to front when logic, whose concern is correct thinking and *which is also the foundation of arithmetic*, borrows its signs from arithmetic. To such a person it will seem more appropriate to develop for logic its own signs, derived from the nature of logic itself; we can then go on to use them throughout the other sciences wherever it is a question of preserving the formal validity of a chain of inference." (Frege, 1881, p. 13; 12, author's emphasis)

This is the only general commitment to a logicist position that can be found in Frege's papers written between 1879 and 1882, after *Begriffsschrift* was completed.

According to Frege, the main task of a *lingua characterica* consists in expressing in a rigorous and unambiguous way the content of scientific disciplines, i.e., to render both the basic components and the logical relations that constitute the complex notions of the discipline. While Schröder considered providing the logic with a set of basic concepts to be indispensable, Frege devised a language which was capable of adopting the symbols of the basic notions of any scientific discipline with a regimented language. In this sense, the realisation of Leibniz's ideal of a *lingua characterica* meant for Frege the application of the formal resources of the concept-script to the basic components of the language of any scientific discipline.

From Frege's perspective, Boolean logic was intended for the study of the abstract properties of the calculus of classes and the sentential calculus. It was not devised to express the specific meaning of the statements of a discipline and, in particular, it was not adequate—at least as of 1880–1882—to capture the process of concept formation. Frege concluded that Boolean logic was an abstract logic and, in this sense, closer to the notion of a calculus.

Although expressive power was a key element in Frege's critique of Boolean logic, the fact that in 1880–1882 this logic did not have a quantification theory was not of great moment. After Peirce's introduction of the relation of subsumption, of individual variables and the interpretation of the generalised sum and product as the quantifiers, Boolean logic exhibited an expressive power comparable to that of the concept-script. However, Schröder never tried to develop a full *calculus ratiocinator*: neither the algebra of absolute terms, the theory of relatives nor the calculus of relatives had a formal apparatus and, specifically, a complete system of inference rules, that could satisfy Frege's demands of rigour in a formalised proof.

After the publication of the third volume of *Vorlesungen*, Schröder had all the technical elements necessary to raise the metalogical questions upon which Löwenheim's contributions in logic were focussed. However, Schröder failed to see the convenience of adopting a model-theoretic point of view such as Löwenheim's and failed to consider substantial questions about the models of the logical fragment of the theory of relatives. I argued that this omission can be understood by exploring Schröder's aim of creating a pasigraphy and the reformulation of mathematics not with any reference to objects but only in terms of relatives.

In contrast, the application of the concept-script to a scientific discipline was not devised by Frege as a process of reduction. Crucially, the meaning of the letters in this formal system is adapted to the presence of a specific domain of entities. In this context, universality is not tied to logic in any substantive sense—at least not as van Heijenoort claimed (1967b, pp. 440–442). In their evaluation of the rival logic, neither Frege nor Schröder opposed the universality of logic and the presence of a limited universe of discourse. All in all, the particular way in which the concept-script was used as a tool for the rigourisation of scientific disciplines shows that Frege did not equate the creation of this formal system as a *lingua characterica* with the logicist programme.

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