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Frege, Peano and the construction of a logical calculus

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Abstract

In contemporary historical studies Peano is usually linked to the logical tradition pioneered by Frege. In this paper I question this association. Specifically, I claim that Frege and Peano developed significantly different conceptions of a logical calculus. On the one hand, I defend that while Frege put the systematisation of the notion of inference at the forefront of his construction of an axiomatic logical system, Peano modelled his early logical systems as mathematical calculi and did not really attempt to justify reasoning. On the other hand, I argue that in later works on logic Peano advanced towards a deductive approach that was closer to Frege's standpoint.

Keywords: Frege · Peano · inference · deduction · logical calculus

1 Introduction

Although Peano never defended a logicist position, he is usually linked to the logical tradition pioneered by Frege[1]. As a key influence for Russell's logic, Peano's mathematical logic is considered to be part of a tradition that is in many respects opposed to the algebra of logic tradition.

In contemporary historical studies, the relation between Frege's and Peano's conceptions of logic has received little attention. In this paper I shall question the inclusion of Peano in the logical tradition which Frege started. Specifically, I shall study Frege's and Peano's conceptions of an axiomatic logical calculus and conclude that they developed substantially different accounts. I shall claim, on the one hand, that while Frege put the systematisation of the notion of inference at the forefront of his construction of an axiomatic logical system, Peano modelled his early logical systems as mathematical calculi and did not really

[1]See, for instance, (Jourdain 1914, p. 14) or (van Heijenoort 1967*a*). This association is still present in recent works on the history of logic. See, for instance, (Haaparanta 2009, pp. 6–7).

attempt to justify reasoning. On the other hand, I shall defend that in later works on logic Peano advanced towards a deductive approach that was closer to Frege's standpoint.

This paper is in four parts. First, I shall consider Frege's notion of an inferential calculus and explain how he realised this notion in two steps. Second, I shall characterise Peano's notion of an axiomatic system. Third, I shall study Peano's early attempts to develop a logical calculus. Last, I shall discuss the evolution of Peano's notion of calculus. I shall divide Peano's works on logic into two periods and characterise the progress from one period to the next as a process of development of the deductive capabilities of logical calculi.

2 Frege's logical calculi

In the Foreword to the first volume of his *magnum opus*, *Grundgesetze der Arithmetik* (1893, 1903) (hereinafter, *Grundgesetze*), Frege connected his approach to the construction of a logical axiomatic system with a mathematical tradition that goes as far back as Euclid and, at the same time, he stressed the novelty of his own approach[2]:

> "The ideal of a rigorous scientific method for mathematics that I have striven to realise here, and which could be named after Euclid, can be characterised as follows. It cannot be required that everything be proven, as this is impossible; but it can be demanded that all propositions appealed to without proof are explicitly declared as such, so that it can be clearly recognised on what the whole structure rests. One must strive to reduce the number of these fundamental laws as far as possible by proving everything that is provable. Furthermore, and in this I go beyond Euclid, I demand that all modes of inference and consequence which are used be listed in advance. Otherwise compliance with the first demand cannot be secured. This ideal I believe I have now essentially achieved." (Frege 1893, p. vi)[3]

Frege's methodology for logic was connected with the axiomatisation in mathematics. The notion of axiomatic system can be associated with a specific conception of calculus, to which I shall refer as 'mathematical calculus'. A mathematical calculus is a system composed of mathematical objects (numbers,

[2]See a similar passage in (Frege 1897, pp. 362–363; 235–236).

[3]Quotes are taken from the English translation or the most recent edition listed in the bibliography. When an English translation is quoted, two page numbers – separated with a semicolon – are given: the first corresponds to the source and the second to the English translation (unless they coincide). When no English translation is available, quotes and page numbers are taken from the source and translated by the author.

manifolds, areas, infinitesimals, etc.) and a set of operations that regiment the relations between these objects. One of the main aspects of Euclid's tradition is the axiomatisation of laws governing the operations of the calculus.

Frege's groundbreaking departure from the mathematical understanding of a calculus consisted in taking judgements as objects of the calculus and providing a complete and explicit system of inference rules which worked as specific operations between these judgements. In Frege's logical system – the concept-script – reasoning was reconstructed as a derivation (*Ableitung*), a succession of judgements that were connected by means of inference rules. This was the basis for the development of an inferential calculus.

Frege's process of construction of a logical calculus was complex. He presented the first modern formal system in *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (1879) (hereinafter, *Begriffsschrift*). *Begriffsschrift*'s concept-script could be seen as a theory of inference. Even the language of the concept-script was conceived with the notion of inference in mind[4]. In Frege's words:

> "[I]ts [the concept-script's] chief purpose should be to test in the most reliable manner the validity of a chain of reasoning and expose each presupposition which tends to creep in unnoticed, so that its source can be investigated. For this reason, I have omitted the expression of everything which is without importance for the chain of inference [*Schlussfolge*]." (Frege 1879, p. iv; 104)

In order to systematise the notion of inference, Frege constructed the calculus of the concept-script as a set of axioms, which he called 'basic laws', and inference rules. He listed eight basic laws (1879, p. 26; 136). These were judgements which were left unproved in *Begriffsschrift*, and which were used for the proof of logical laws. Frege appealed to the fact that they constituted the kernel of content of the concept-script (1879, p. 25; 136). Even though Frege acknowledged that his specific choice of basic laws could be made otherwise, he also made it clear that his choice was methodologically motivated.

The only inference rule that Frege introduced explicitly as such is Modus Ponens (hereinafter, [MP]) (1879, pp. 7–9; 117–120). However, he used several times in the proofs contained in Chapter III of *Begriffsschrift* two additional inference rules[5]. Frege presented [G] and [C] in the exposition of the language

[4]Frege explicitly claimed that in *Begriffsschrift* "the only thing considered in a judgement is that which influences its *possible consequences* [*Folgerungen*]" (1879, §3, p. 3; 113), and he called this its '*conceptual content*'.

[5]Let A be a formula and $\Phi(a)$ a formula in which the letter a occurs. For the sake of clarity, I formulate all concept-script propositions in a hybrid notation. All content strokes – save that which

of the concept-script, but he neither characterised them as inference rules nor included them explicitly in the calculus. On those occasions in which a proof required the application of one of these rules, he did not mention them; their application was left to be guessed by the reader[6]. More importantly, although Frege offered some remarks regarding the notion of substitution, he did not define substitutions and, crucially, he did not characterise the corresponding inference rules. Specifically, according to contemporary standards of rigour, Frege should have defined substitutions for propositional variables, of individual terms for individual variables, of relation symbols for predicate variables, and finally of formulas for predicate variables. Each kind of substitution requires an inference rule that permits the derivation of an instance of substitution from a theorem of the calculus. Virtually all proofs contained in *Begriffsschrift* require the application of one or more substitution rules[7].

The inferential character of the calculus of *Begriffsschrift* is exemplified in the proofs contained in Chapters II and III of this book. Consider, for instance, the derivation of Proposition (62) (Frege 1879, p. 52; 164):

1. $\vdash (d \to (b \to a)) \to (b \to (d \to a))$, Prop. (8).
2. $\vdash [\forall \mathfrak{a}(g(\mathfrak{a}) \to f(\mathfrak{a})) \to (g(x) \to f(x))] \to [g(x) \to (\forall \mathfrak{a}(g(\mathfrak{a}) \to f(\mathfrak{a})) \to f(x))]$, Subst. in (1):

follows the judgement stroke – are eliminated and the generality symbols $\smile^{\mathfrak{a}}$ and the connectives (negation and conditional) are rendered according to their contemporary equivalents. The different typefaces used by Frege are maintained.

I shall call Generalisation (hereinafter, [G]) the first of the aforementioned additional inference rules. Frege introduced [G] thus (1879, p. 21; 132):

$$\frac{\vdash \varPhi(a)}{\vdash \forall \mathfrak{a}\, \varPhi(\mathfrak{a}),}\ [\mathrm{G}]$$

if a does not occur in $\forall \mathfrak{a} \varPhi(\mathfrak{a})$.

I shall call Confinement of the quantifier (hereinafter, [C]) the following rule (1879, p. 21; 132):

$$\frac{\vdash A \to \varPhi(a)}{\vdash A \to \forall \mathfrak{a}\, \varPhi(\mathfrak{a}),}\ [\mathrm{C}]$$

if a does not occur either in A or in $\forall \mathfrak{a} \varPhi(\mathfrak{a})$.

[6]In the Preface on *Begriffsschrift* Frege appealed to methodological reasons to restrict the list of inference rules of the concept-script to [MP] (1879, p. vii; 107). However, Frege also acknowledged the possibility of adding derived rules to the calculus to simplify proofs. Moreover, he would qualify his claim on the limitation of inference rules in the Preface by stating that [MP] is the only inference rule of the concept-script that allows the derivation of a conclusion from several judgements. Any other inference rule used in the concept-script – specifically, [G] and [C] – allows the derivation of a judgement from a single judgement.

[7]The question whether Frege did include or should have included one or several substitution rules in *Begriffsschrift* is complex and far surpasses the scope of this paper. With Calixto Badesa, I offered an analysis of this issue in (Badesa and Bertran-San Millán 2020).

$$\begin{array}{ccc} a & b & d \\ \hline f(x) & g(x) & \forall \mathfrak{a}(g(\mathfrak{a}) \rightarrow f(\mathfrak{a})) \end{array}$$

3. $\vdash \forall \mathfrak{a} f(\mathfrak{a}) \rightarrow f(c)$, Prop (58).

4. $\vdash \forall \mathfrak{a}(g(\mathfrak{a}) \rightarrow f(\mathfrak{a})) \rightarrow (g(x) \rightarrow f(x))$, Subst. in (3):

$$\begin{array}{cc} f(A) & c \\ \hline (g(A) \rightarrow f(A)) & x \end{array}$$

5. $\vdash g(x) \rightarrow (\forall \mathfrak{a}(g(\mathfrak{a}) \rightarrow f(\mathfrak{a})) \rightarrow f(x))$, Prop. (62): [MP] (2),(4).

All proofs of *Begriffsschrift* shared the structure of this derivation. First, Frege indicated two logical principles that were used as premises. The premises of this derivation are Propositions (8) and (58), which are basic laws of the calculus. Then, Frege listed the substitutions that have to be performed to the premises[8]. In this way, instances of the premises were inferred. Sometimes, [G] or [C] were applied to one or both premises. Finally, [MP] was applied to the respective instances of the premises in order to obtain the conclusion.

Frege's stress on a systematisation of the notion of inference was further developed in *Grundgesetze*. In this work he offered a revised version of the concept-script. In *Grundgesetze* Frege inverted the strategy adopted in *Begriffsschrift* and reduced the list of basic laws whilst he considerably enlarged the list of inference rules. The concept-script of *Grundgesetze* was composed of six basic laws (1893, p. 61) – two of which, basic laws (V) and (VI), involve the notion of value-range, which did not appear in *Begriffsschrift*.

In contrast with how the rules [G] and [C] were treated in *Begriffsschrift*, Frege offered an exhaustive list of inference rules in *Grundgesetze*. He provided twelve inference rules (1893, pp. 61–63), although some of them should be considered derived rules and others should be taken as instantiation principles. As a result, Frege obtained a formal system whose derivations were completely regimented by the inference rules. No formal step was left implicit and no appeal to non-defined logical principles needed to be made.

3 Peano's axiomatic systems

Peano presented his first axiomatisation of arithmetic in *Arithmetices principia nova methodo exposita* (1889) (hereinafter, *Arithmetices principia*)[9]. Peano's approach of providing a set of axioms for arithmetic fits with the general trend in

[8]The inference from premises to instances of substitution has been represented as steps (2) and (4) in the derivation, where substitution tables have been included. Note that the upper row in the table contains the letters that are to be replaced in the premises and the lower row their instances of substitution.

[9]Peano's axiomatisation of arithmetic is equivalent to Dedekind's, which was included in *Was sind und was sollen die Zahlen?* (1888). In *Arithmetices principia* Peano only said in connection to

nineteenth-century mathematics of establishing the basic principles of a mathematical theory[10].

Peano's notion of an axiomatic theory relied on the establishment of two privileged sets of elements: the notions that remain undefined, which are considered simple and from which all ideas of the theory can be defined, and the axioms, that is, those propositions that receive no proof and from which all theorems are obtained. In *Arithmetices principia* Peano applied this conceptual framework to his presentation of arithmetic:

> "Those arithmetical signs which may be expressed by using others along with signs of logic represent the ideas that we can define. Thus I have defined every sign, if you except the four which are contained in the explanations of §1 [N, 1, $+1$, $=$]. If, as I believe, these cannot be reduced further, then the ideas expressed by them may not be defined by ideas already supposed to be known.
>
> Propositions which are deduced from others by the operations of logic are *theorems*; those for which this is not true I have called *axioms*. There are nine *axioms* here (§1), and they express fundamental properties of the undefined signs." (Peano 1889, pp. iii-iv; 102)

In 'Formole di Logica Matematica' (1891*a*, pp. 24–25) Peano developed this approach and introduced the distinction between primitive and derived ideas and propositions. Similarly to Frege, he acknowledged a degree of arbitrariness in the selection of primitive propositions and also of primitive notions (1891*a*, p. 26, fn. 1).

As a means of relieving arithmetic of the use of natural language and thus eliminating all the ambiguities and inaccuracies introduced by the use of words in the expression of mathematical concepts and proofs, Peano complemented the basic linguistic elements of arithmetic with the formal resources provided

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Dedekind (1888) that it was "quite useful" (1889, p. v; 103) to him and made a terminological remark in which he alluded to Dedekind. However, in 'Sul concetto di numero' (1891*c*, p. 93), Peano stated that his axioms of arithmetic were due to Dedekind. Later in this work Peano made the following remark:

> "Here (Peano 1891*c*) the number is not defined, but its fundamental properties are stated. Dedekind defines the number instead, and specifically calls number what satisfies the aforementioned conditions. Evidently the two things coincide." (Peano 1891*c*, p. 94)

[10] As Grattan-Guinness (2011, p. 135) claimed, a probable source for Peano's vindication of rigour was the process of the arithmetisation of analysis, which had Weierstrass as one of its main proponents. Von Plato (2017, pp. 40–49, 54–57) has convincingly explained the influence of Grassmann's definitions by recursion in Peano's presentation of arithmetic. In *Arithmetices principia* Peano explicitly acknowledged (Grassmann 1861) as a source for his proofs of arithmetic. See also (Lolli 2011).

by logic. This complementation required the development of a logical language and the establishment of the principles of logic. The use of logical language and, specifically, logical connectives and quantification, made possible, one the one hand, to formally define arithmetical notions, and on the other hand to express the laws of arithmetic. On these grounds arithmetic could be formally constructed as an axiomatic theory.

Peano used as a basis for his mathematical logic the calculus of classes and the calculus of propositions developed by algebraic logicians[11]. However, unlike algebraic logicians, Peano was reluctant to use arithmetical symbols in order to express logical notions or relations between classes. He wanted to preserve the specific meaning of arithmetical symbols and, at the same time, avoid the confusions that would arise had they acquired, in addition to their mathematical meaning, a logical meaning. In *Arithmetices principia* Peano provided separate lists of the symbols of logic and the symbols of arithmetic (1889, p. vi; 103). There is only one symbol which appeared in both lists, namely '='. It was first presented as a logical symbol expressing equality between propositions (1889, p. viii; 105) and classes (1889, p. xi; 108). Then, the symbol of equality was introduced as an arithmetical symbol and Peano warned about its ambiguity: "= means *is equal to* (this must be considered as a new sign, although it has the appearance of a sign of logic)" (1889, p. 1; 113)[12].

One of the most significant elements of Peano's axiomatisation of arithmetic is the clear separation of logical principles and arithmetical axioms[13]. In this regard, he departed from the algebra of logic tradition. Peano considered the principles of the calculus of classes to be part of logic, but he still distinguished

[11]Peano explicitly acknowledged that his logic was developed from the presentations of Boole, Peirce and Schröder. See (Peano 1888, p. vii, fn. 7) and (Peano 1889, p. iv; 86, fn.). The influence of Schröder is particularly clear in Peano's first presentation of logic, contained in (1888, pp. 1–20). See also (Grattan-Guinness 2011, pp. 136–139).

[12]Frege criticised several times Peano's piecemeal definition of equality. See (Frege 1976, pp. 181–185) and (Frege 1897, pp. 366–367; 238).

See also (Peano 1891*b*, pp. 9–10; 158), where Peano made explicit remarks about the separation of the symbols of arithmetic and those of logic. Another example is Peano's introduction of the symbol 'Λ', used to denote both the empty class and the absurd. In (1897*a*) Peano explicitly rejected to use an arithmetical symbol such as '0' for this purpose:

> "The sign Λ is the reversed initial of the word *true* [original French *vrai*] [...]. Several Authors, especially Boole and Mr. Schröder indicate the null class with the symbol 0; notation that we had to abandon so as not to confuse, in our formulas, the logical symbols and the algebraic ones." (Peano 1897*a*, p. 46)

[13]In *Arithmetices principia* Peano included four axioms in the list of the axioms of arithmetic that expressed the basic properties of the relation of equality (1889, p. 1; 113). These axioms characterised equality as a logical relation: an equivalence relation that also preserves the principle of substitutivity of identicals. The four axioms involving equality were detached from the list of the axioms of arithmetic and included among the logical principles in later presentations. See (Peano 1891*c*, p. 90) and (Peano 1898*a*, p. 1).

them from the principles of sentential logic.

4 Peano's early attempts to develop a logical calculus

In published work and personal correspondence Frege commented on Peano's works[14]. The main focus of Frege's remarks was Peano's notion of definition[15]. In 'Über die Begriffsschrift des Herrn Peano und meine eigene' (1897) Frege discussed in detail the basic components of Peano's presentation of his mathematical logic. Frege took *Notations de logique mathématique* (Peano 1894) and the first volume of *Formulaire de mathématiques* (Peano 1895*a*) (hereinafter, *Formulaire*) as representative of Peano's position.

In his paper on Peano's mathematical logic Frege failed to acknowledge the importance of Peano's axiomatisation of arithmetic. Most likely Frege's logicist thesis played a role in his account of Peano's axiomatisation of arithmetic. According to Frege's logicism, if arithmetic has a set of axioms, they are ultimately logical laws. In 'Über formale Theorien der Arithmetik' Frege stressed the differences between geometry and arithmetic by appealing to the notion of axiom:

> "Herewith arithmetic is placed in direct contrast with geometry, which, as surely no mathematician will doubt, requires certain axioms peculiar to it where the contrary of these axioms – considered from a purely logical point of view – is just as possible, i.e., is without contradiction." (Frege 1885, p. 94; 112)

Even if the axioms of arithmetic were, in fact, logical laws for Frege, it was still a mathematically remarkable fact that all theorems of arithmetic could be derived from five axioms (i.e., nine minus the axioms that involve logical equality – see Footnote 13). This was actually connected to one of Frege's methodological principles[16].

[14]Peano then published part of this correspondence in *Revue de mathématiques*, of which he was the editor. See (Frege 1896) and (Peano 1898*b*).

[15]See, for instance, (Frege 1903, pp 70–71, fn.).

[16]Frege did not comment – at least in what has been preserved in his *Nachlaß* – on the importance of Dedekind's definition of the system of natural numbers either. In fact, Frege's remarks about Dedekind (1888) in *Grundgesetze* are very similar in tone to his criticism of Peano (1893, pp. vii–viii). It is not casual that Frege criticised both Dedekind's and Peano's presentations of arithmetic on methodological grounds and, specifically, for their lack of a fully formalised characterisation of the notion of proof. To rigorously settle the foundations of the laws of arithmetic was for Frege the core of his endeavour in *Grundgesetze*, and the development of an inferential calculus was instrumental for this aim.

Moreover, Frege praised the expressive capabilities of Peano's logical language and contrasted it with those of Boole and Schröder – which, from Frege's perspective, were inadequate tools for the expression of arithmetical truths (1897, pp. 370–371; 242). At the same time, Frege criticised Peano's failure to provide a full inferential calculus:

> "In any case, less emphasis is placed upon strictness in conducting a proof, and upon logical perfection, here [in (Peano 1895*a*)] than in my conceptual notation [...]. That the conduct of proof is thrust into the background here [in (Peano 1895*a*)] is due also to the absence of rules of inference, for the formulae in Part I of *Formulaire* can offer no substitute for them. The question here is simply how, from one of those formulae, or from two of them, a new one is obtained." (Frege 1897, pp. 366–367; 238)

According to Frege, Peano could not guarantee a fully rigorous treatment of arithmetic if he did not provide the means to formalise proofs. All in all, Frege was stressing the difference between a mathematical calculus and an inferential calculus, and accusing Peano of not having fulfilled his own demands of rigour by failing to provide a complete system of rules of inference.

Two years before the publication of Frege's 1897 paper, Peano published a review to the first volume of *Grundgesetze*. Among other things, he mentioned Frege's stress on the rules of inference and offered the following critical remarks:

> "The author [Frege] shows great concern over his rules of reasoning, which he explains in ordinary language. When translated into symbols, these become logical identities, all of which are contained in Part I of *Formulaire* [...]. The only work that can be done on these rules of reasoning is that of discovering whether one rule is the equivalent of a whole combination of others; and thus, continuing this analysis, one will arrive at the system of the most simple rules, which in *Formulaire* Part I are called primitive propositions." (Peano 1895*b*, p. 127; 31)

This quotation captures Peano's early position concerning logical proofs[17]. He defended that the principles of reasoning and, specifically, the inference rules, are expressed by laws of logic, which capture the deductive aspect of a calculus of logic. This was supported by his interpretation of '⊃', a primitive symbol of Peano's mathematical logic. Peano wanted to take advantage of the parallelism between the logic of classes and sentential logic and used '⊃' to

[17] See also (Peano 1894, p. 51), where Peano identified rules of reasoning with logical propositions and considered the former to be analogous to laws of algebra.

express the relation of inclusion and the conditional, respectively. Moreover, although he realised that the relation between the antecedent and the consequent of a conditional is presented informally as a different relation to that between premises and consequence in a logical derivation (properly, the relation of deducibility), he still assumed that both relations could be conveniently expressed using the same symbol. In Peano's words:

> "The sign $\supset$ between classes may be read 'is contained in', while between propositions it is read 'we deduce'. The fact that it may be read in several ways does not prove that it has several meanings, but only that ordinary language has several terms to represent the same idea. The term which would better correspond to the sign $\supset$ in its various positions could possibly be 'hence', 'ergo'." (Peano 1896–1897*b*, p. 573; 197)

According to this interpretation of '$\supset$', Peano defended that no inference rule was required if its content was expressed by the laws of logic and, ultimately, by the primitive propositions of the calculus, i.e., the axioms of logic.

Peano's view on the inference rules of a logical calculus is connected with a polemic that has aroused in contemporary historical studies. In the Introduction to his translation of *Arithmetices principia*, van Heijenoort claims that in the logical calculus presented by Peano "[t]here is [...] a great defect. The formulas are simply listed, not derived, because no rules of inference are given" (1967*b*, p. 84)[18]. In contrast, von Plato (2017, pp. 50–57) argues that Peano's derivations in *Arithmetices principia* are fully formal and consist in the successive application either of Modus Ponens or instantiation[19]. Von Plato's account focuses on the derivation of Theorem (11) (Peano 1889, pp. 2; 113–114):

$$2\,\varepsilon\,\mathrm{N} \tag{11}$$

Proof:

$\mathrm{P1}.\supset:$	$1\,\varepsilon\,\mathrm{N}$	(1)
$1[a](\mathrm{P6}).\supset:$	$1\,\varepsilon\,\mathrm{N}\,.\supset.\,1+1\,\varepsilon\,\mathrm{N}$	(2)
$(1)(2).\supset:$	$1+1\,\varepsilon\,\mathrm{N}$	(3)
$\mathrm{P10}.\supset:$	$2=1+1$	(4)
$(4).(3).(2,1+1)[a,b](\mathrm{P5}):\supset:$	$2\,\varepsilon\,\mathrm{N}$	(Theorem)

Clearly, lines (1), (2) and (4) consist in the instantiation of axioms P1, P6 and definition P10, respectively. Substitutions are indicated with the following notational system: $(x,y,z)[a,b,c]\phi$ means that a, b and c are replaced with x, y and

[18] (Goldfarb 1980, p. 179) and (Grattan-Guinness 2000, p. 228) defend similar positions.
[19] See also (von Plato 2014, pp. 243–246) and (von Plato 2018).

z, respectively, in formula ϕ. Line (3) consists in the application of Modus Ponens to the results of (1) and (2). Even though Peano affirmed in his review to *Grundgesetze* (1895*b*) that the principles of reasoning were incorporated in the calculus as logical laws, in *Arithmetices principia* he neither mentioned Modus Ponens as an inference rule nor included a corresponding logical law:

$$a\,.\,a \supset b : \supset : b$$

in the list of propositions of logic (1889, pp. viii-ix; 105–106)[20]. However, line (5) is not the result of the application of Modus Ponens to what has been deduced in previous lines of the derivation. In (3) and (4), Peano obtained $1 + 1\,\varepsilon\,\mathrm{N}$ and $2 = 1 + 1$ *separately*. In full rigour, he could not use these formulas and the following instance of (P5):

$$2 = 1 + 1\,.\,1 + 1\,\varepsilon\,\mathrm{N} : \supset .\, 2\,\varepsilon\,\mathrm{N}$$

to apply Modus Ponens, for he did not deduce the antecedent of this formula, $2 = 1 + 1.1 + 1\,\varepsilon\,\mathrm{N}$, i.e., the conjunction of $2 = 1 + 1$ and $1 + 1\,\varepsilon\,\mathrm{N}$. In order to solve this issue, Peano would need a rule of introduction of the conjunction (such as $\{p, q\} \vdash p \wedge q$), which could not be expressed in the language of his mathematical logic as a logical law[21].

More importantly, several theorems require a conditional proof. Let us consider an example. Peano divided Theorem (13):

$$a, b, c, d\,\varepsilon\,\mathrm{N}\,.\,a = b\,.\,b = c\,.\,c = d\,.\supset.\,a = d \qquad (13)$$

[20]As I shall explain below, this logical law was a primitive proposition of the calculus of classes Peano provided in (1891*a*, p. 27).

[21]Incidentally, Peano could have amended the proof of Theorem (11) in such a way that each of its steps consisted either in an instantiation or the application of Modus Ponens. The following derivation can be carried out using, besides Modus Ponens and instantiation, the logical laws explicitly stated by Peano in *Arithmetices principia*:

Let p be $2 = 1 + 1$, q be $1 + 1\,\varepsilon\,\mathrm{N}$ and r be $2\,\varepsilon\,\mathrm{N}$:

$\mathrm{P1}.\supset.$	$1\,\varepsilon\,\mathrm{N}$	(1)
$1[a](\mathrm{P6}).\supset:$	$1\,\varepsilon\,\mathrm{N}.\supset.\,q$	(2)
$(1)(2).\supset.$	$q\ [1 + 1\,\varepsilon\,\mathrm{N}]$	(3)
$\mathrm{P10}.\supset.$	$p\ [2 = 1 + 1]$	(4)
$(p, q, r)[a, b, c](\mathrm{L42}).\supset\therefore$	$p \supset .\, q \supset r :=: pq \supset r$	(5)
$(p \supset .\, q \supset r, pq \supset r)[a, b](\mathrm{L10}).\supset ::$	$p \supset .\, q \supset r :=: pq \supset r \therefore \supset \therefore pq \supset r. \supset : p \supset .\, q \supset r$	(6)
$(5)(6).\supset \therefore$	$pq \supset r. \supset : p \supset .\, q \supset r$	(7)
$(2, 1 + 1)[a, b](\mathrm{P5}).(7) : \supset :$	$p \supset .\, q \supset r$	(8)
$(4)(8).\supset.$	$q \supset r$	(9)
$(3)(9).\supset.$	$r\ [2\,\varepsilon\,\mathrm{N}]$	(Th.)

The fact that this derivation can be completed by appealing only to logical laws included in Peano's list of propositions (besides Modus Ponens and a rule of substitution) is independent of the fact that Peano's derivation needs to appeal to logical principles that he did not explicitly include in the calculus.

into Hypothesis (which corresponds to its antecedent) and Thesis (its consequent). As a means to justify Theorem (13), he provided the schema of a proof that included the Hypothesis as a premise and concluded in the following way:

$$\{a, b, c, d\,\varepsilon\,\mathrm{N} \,.\, a = b \,.\, b = c \,.\, c = d\} \vdash a = d.$$

In order to complete this derivation, a rule of introduction of the conditional would be needed. Again, Peano neither included the corresponding principle in the list of propositions of logic nor could he express such a principle in the language of his mathematical logic.

Even though he explained the notions of theorem and axiom in *Arithmetices principia*, Peano did not isolate a group of primitive propositions for the calculi of classes and propositions[22]. In this work he only axiomatised arithmetic; the principles of logic were listed without any distinction concerning their status in the calculus. In 'Formole di Logica Matematica' Peano offered for the first time a regimentation of the principles of logic and provided a list of primitive propositions (1891*a*, pp. 25–26)[23]. In this short paper Peano included as primitive propositions the following laws (which he called, respectively, P7 and P8) (1891*a*, p. 27):

$$a \,.\, a \supset b : \supset . \, b \tag{7}$$

$$a \supset b \,.\, b \supset c : \supset . \, a \supset c \tag{8}$$

In the proofs of 'Formole di Logica Matematica' (1891*a*) P7 and P8 were used either as logical axioms (i.e., as formulas of the language subject to substitutions) or as principles of reasoning. However, the proofs of 'Formole di Logica Matematica' (1891*a*) also required the application of inference rules that were not even mentioned by Peano: introduction and elimination of the conjunction, and introduction of the conditional. These inference rules systematise argumentation techniques that are either implicit or common in mathematical practice. However, what is at stake here is not the correctness of the proofs included in Peano's early works on logic, but his success in formalising inferences and rigorously justifying logical reasoning.

Peano's aim at expressing all relevant principles of reasoning as laws of logic (leaving aside substitutions and, possibly, Modus Ponens) was legitimate in the

[22]Note that, although Peirce (1880) advanced towards an axiomatisation of the calculus of classes, this calculus was first axiomatised by Schröder in the first volume of *Vorlesungen über die Algebra der Logik* (1890, pp. 168–293) (hereinafter, *Vorlesungen*), one year after the publication of *Arithmetices principia*. An axiomatisation of the calculus of propositions was contained in the second volume of *Vorlesungen* (Schröder 1891, pp. 28–32, 52) and consisted in adding a new axiom to the axioms of the calculus of classes.

[23]The terminology 'primitive proposition' and 'theorem' had been introduced by Peano before 1891 and applied to the principles of arithmetic. See, for instance, (Peano 1889, p. xvi).

context of the construction of a deductive calculus. Had he included in the calculus all logical laws that, once seen as principles of reasoning, were needed in the proofs, his logical calculus could be seen as a deductive calculus and Frege's complaints about the rigour of Peano's proofs would lose power. However, several proofs included in *Arithmetices principia* and 'Formole di Logica Matematica' (1891*a*) require the application of principles of reasoning that neither were included in the calculus as logical laws nor – as I claim – could they be formulated in the language of Peano's mathematical logic.

5 The development of Peano's notion of calculus

The logical system that Peano used as a basis for his early mathematical logic, Schröder's algebra of logic, was not intended for use as a deductive system. Algebraic logicians developed their algebraic theory by means of the kind of reasoning that is used in any mathematical theory (especially, in algebra). Schröder was convinced that a single mathematical theory, the algebra of relatives – which he presented in the third volume of *Vorlesungen* (1895), included the calculus of classes and the calculus of propositions. It is therefore natural that Schröder did not appreciate the convenience of justifying reasoning: he neither isolated the logical principles from the principles of the algebra of relatives nor formally characterised the notion of deducibility by means of a complete system of inference rules[24]. In this regard, Schröder's approach was in contrast with Frege's, for whom a precise and rigorous formulation of the logical principles – basic laws and inference rules – was a key priority both in his early and later presentations of the concept-script.

However, Peano was not interested in studying the algebraic principles of logic, as Schröder did. In fact, Peano disregarded algebraic laws in constructing an axiomatic system of logic. Yet since he developed his logic of classes and sentential logic based upon Boole's, Peirce's and Schröder's calculi of classes and calculi of propositions, he was not pressed to acknowledge that the formalisation of logical reasoning – which was instrumental for obtaining arithmetical and logical theorems – required, besides a specific set of primitive propositions, a complete system of inference rules. With the notion of mathematical calculus in mind, in his early works on logic Peano was convinced that the formulation of logical laws – seen as rules of transformation of formulas – was sufficient to capture the principles of reasoning involved in a derivation. He did not appreciate Frege's inferential approach and failed to recognise that the chains of inference that appeared in his early works on logic were not, as we saw, gapless.

[24]Concerning Schröder's conception of logic and, specifically, his failure to isolate the logical fragment of the algebra of relatives, see (Badesa 2004, pp. 53–65). See also (Bertran-San Millán 2020).

Peano adopted in his presentations of mathematical logic the division – which did not occur in Frege's presentations of the concept-script – between the calculus of classes and the calculus of propositions that characterised the contributions to the algebra of logic tradition. From 1889 to 1891, Peano took the calculus of propositions as primary and the calculus of classes as derivative. However, he inverted this order of precedence from *Notations de Loqique Mathématique* (1894) onwards. In Part I of the second volume of the *Formulaire* (1897*a*) Peano provided an axiomatisation of the calculus of classes and, following Schröder's practice, he did not formulate specific axioms for the calculus of propositions. On the contrary, he used the axioms and propositions of the calculus of classes and interpreted them as if they were formulated for sentential logic[25]. Let us see an example. Peano included the following law:

$$a, b, c, d \,\varepsilon\, \mathrm{K} : \supset : ab \supset c \,.\, ac \supset d \,.\supset.\, ab \supset d. \tag{37}$$

in the calculus of classes (1897*a*, p. 5). As the antecedent $a, b, c, d \,\varepsilon\, \mathrm{K}$ shows, the letters of this proposition are interpreted as classes and thus the second, third and fifth occurrences of the symbol '$\supset$' have to be interpreted as inclusion between classes. However, in the proof of Proposition (505) (1897*a*, p. 16), proposition (37) is used as a means to obtain a formula of the form $pq \supset t$ from the formulas $pq \supset r$ and $pr \supset t$, which requires to employ (37) as if it were formulated in the calculus of propositions:

$$pq \supset r \,.\, pr \supset t \,.\supset.\, pq \supset t,$$

where p, q, r and t are propositional letters and the first, the second and the fourth occurrences of the symbol '$\supset$' must be interpreted as the conditional[26].

Related to this, another prominent shortcoming for the development of a deductive calculus in Peano's mathematical logic was his treatment of the symbol '$\supset$'. Frege wrote a lengthy comment on Peano's use of this symbol, criticising the latter's decision to attach different meanings to it (1897, pp. 371–373; 243–244). He distinguished four different uses of '$\supset$': the relation of inclusion between

[25] On Schröder's interpretation of the axioms of sentential logic and their use in proofs see (Badesa 2004, p. 26).

[26] Peano's Proposition (505) is the following:

$$u \,\varepsilon\, a \,\mathfrak{z}\, b \,.\, c \supset a \,.\supset.\, u \,\varepsilon\, c \,\mathfrak{z}\, b, \tag{505}$$

where the Hp (or Hyp in other texts) is $u \,\varepsilon\, a \,\mathfrak{z}\, b \,.\, c \supset a$. The relevant step – for our present purposes – in the proof of (505) involves as premises:

1. $\mathrm{Hp} \,.\, x \,\varepsilon\, c \,.\supset.\, x \,\varepsilon\, a.$
2. $\mathrm{Hp} \,.\, x \,\varepsilon\, a \,.\supset.\, xu \,\varepsilon\, b.$

Peano proposed to apply (37) as a rule of reasoning in order to obtain:

3. $\mathrm{Hp} \,.\, x \,\varepsilon\, c \,.\supset.\, xu \,\varepsilon\, b.$

According to my interpretation, $p = \mathrm{Hp}$ (i.e, $p = u \,\varepsilon\, a \,\mathfrak{z}\, b \,.\, c \supset a$), $q = x \,\varepsilon\, c$, $r = x \,\varepsilon\, a$ and $t = xu \,\varepsilon\, b$.

classes, conditional, generalised conditional and the relation of deducibility. Frege was aware that the association of several meanings to '$\supset$' was not a matter of confusion. However, the economy of symbols that the use of '$\supset$' implied was for Frege a poor advantage if it diminished the clarity of the symbolism. The aforementioned use of (37) in the proof of Proposition (505) is an indication that the ambiguity between different interpretations of '$\supset$' was not always justified. Furthermore, the association of the conditional and the relation of deducibility was for Frege an indication of Peano's difficulties in recognising the convenience of developing logic as a theory of inference.

Peano progressively modified the presentation of his mathematical logic and began to systematise the relation of deducibility in his later works on logic. In Part I of the second volume of the *Formulaire* he provided a list of inference rules (1897*a*, p. 34), which were identified with laws of the calculus of classes (and on some occasions were applied, as we have seen, as if they were formulated for sentential logic). Two years later, in Part III of the second volume of the *Formulaire*, Peano almost copied his presentation of the rules of reasoning from Part I (Cfr. (1897*a*, p. 2)), but enlarged it with significant additions (1899, p. 11). First, although he still identified the rules of reasoning with specific laws of the calculus of classes, he explicitly acknowledged that former expressed general deductive relations that could be applied either to the the calculus of classes or to the calculus of propositions. Second, he provided an informal formulation of the following principle of reasoning: "[o]ne can unite true P [propositions] to a Hyp [hypothesis], in the order one pleases, without changing the value of the Hyp" (1899, p. 11). Since in conditional proofs the antecedent of the conditional to be proved, i.e., its hypothesis, was used as a premise, this rule can be seen as an introduction of the conjunction. The fact that Peano did not identify a specific law of logic with this principle of reasoning can be seen as an implicit acknowledgement that it played a role in the calculus but could not be expressed in its language (see Section 4).

A further step was made in 'Formules de Logique Mathématique' (1900). In this paper, Peano again provided a list of inference rules, but this time presented them as general rules of reasoning, not tied to a specific calculus. He then associated each rule with an analogous rule specific for the calculus of classes, which in turn was identified with a logical law. As a result, the calculi of classes and propositions had a common set of inference rules, and in addition the former had a specific set of derived rules – based on logical laws. See, for instance, Peano's presentation of the rule he labelled 'Syllogism':

> "Let p, q, r, s be propositions.
> Syll, abbreviation of Syllogism, indicates the form
>
> $$p \supset q \,.\, q \supset r \,.\supset .\, p \supset r.$$

> If the propositions are reduced to the form $x \varepsilon a$, where a is a Cls, the syllogism is expressed by the P2 · 4 $[a, b, c \,\varepsilon\, \text{Cls} \,.\, a \supset b \,.\, b \supset c \,.\, \supset .\, a \supset c]$. But we will apply Syll even when it is about P [propositions] not yet reduced to the form $x \varepsilon a$." (Peano 1900, p. 14)

Note that Peano's approach still retained some of the ambiguity expressed by the symbol '$\supset$'. Specifically, unless

$$p \supset q \,.\, q \supset r \,.\, \supset .\, p \supset r.$$

is read as an expression of the calculus of propositions (which would entail that it no longer expresses the general form of an inference rule), the third occurrence of '$\supset$' corresponds to the deducibility symbol, while the remaining occurrences correspond to conditionals[27].

This kind of presentation can be taken as a sign that Peano had abandoned the practice of using laws of the calculus of classes as general rules of reasoning, as if they were formulated also for the calculus of propositions. This was instrumental for a full formalisation of proofs. In fact, even though Peano still provided sketchy indications of most proofs, which included only the most prominent principles and inference rules involved, in 'Formules de Logique Mathématique' he included the proof of a theorem of the logic of classes:

$$a, b, c \,\varepsilon\, \text{Cls}.\, b \supset c \,.\, \supset .\, ab \supset ac \tag{3·5}$$

which contained all formal steps, and he also gave explicit indications of the application of the inference rules (1900, pp. 17–18). The comparison between this proof and those of *Arithmetices principia* is a good witness of the evolution of Peano's notion of calculus and the progress towards a proper deductive approach.

6 Concluding remarks

Despite the consensus regarding the association of Peano with the logical tradition initiated by Frege, there are some elements in Peano's mathematical logic that show a departure from core aspects of Frege's conception of logic. In this paper I focused on one specific topic in Frege's and Peano's contributions to logic: the notion of logical calculus. I contrasted their views on this topic, concluding that they developed significantly divergent conceptions.

Frege was pioneer among modern logicians in creating a logical calculus that could be seen as a theory of inference. His insistence on the formalisation of the logical transformations performed in a derivation put him in stark

[27] A similar ambiguity can be found in Schröder's presentation of the calculus of propositions in the second volume of *Vorlesungen* (1891, pp. 28–32, 52). See (Badesa 2004, pp. 25–30).

contrast with the methodology of algebraic logicians, who maintained a mathematical notion of calculus. Peano's early works on logic exhibited the proximity of his notion of axiomatic calculus with that of algebraic logicians, particularly Schröder. Although Peano disregarded the algebraic elements of Schröder's mathematical theory of logic, he did not immediately adopt a deductive approach by means of which he could fully regiment the proofs of his mathematical logic. I showed that some of the chains of inferences included in Peano's early works on logic were not completely free of gaps. Moreover, I argued that Peano's association of the conditional with the notion of logical consequences showed his difficulties in fully overcoming Schröder's algebraic approach to logic.

Peano's notion of logical calculus did not remain static. In his later works on logic, he isolated for the first time a set of inference rules and stopped seeing them merely as reformulations of laws of the calculus of classes. I have defended that this was instrumental for the development of a fully formalised concept of proof, which in turn showed Peano's progress towards a deductive notion of calculus.

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References

Badesa, C. (2004), *The Birth of Model Theory*, Princeton University Press, Princeton.

Badesa, C. and Bertran-San Millán, J. (2020), *Begriffsschrift*'s logic. Forthcoming in *Notre Dame Journal of Formal Logic*.

Bertran-San Millán, J. (2020), '*Lingua characterica* and *calculus ratioctinator*: The Leibnizian background of the Frege-Schröder polemic', *Review of Symbolic Logic* pp. 1–35.

Dedekind, R. (1888), *Was sind und was sollen die Zahlen?*, Friedrich Vieweg, Braunschweig.

Dudman, V. H. (1971), 'Peano's review of Frege's *Grundgesetze*', *Southern Journal of Philosophy* **9**, 25–37.

Frege, G. (1879), *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Louis Nebert, Halle. Reedition in Frege (1964), pp. 1–88. English translation by T. W. Bynum in (Frege 1972, pp. 101–203).

Frege, G. (1885), 'Über formale Theorien der Arithmetik', *Jenaische Zeitschrift für Naturwissenschaft* **19**, 94–104. English translation by E. H. Kluge in (Frege 1984, pp. 112–121).

Frege, G. (1893), *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*, Vol. I, Hermann Pohle, Jena. English translation by P. Ebert and M. Rossberg in (Frege 2013).

Frege, G. (1896), Lettera del Sig. G. Frege all'Editore. Letter to G. Peano dated August 29, 1896. Published in 1898 in *Revue de Mathématiques* **6**, pp. 53–59, Reprinted in (Frege 1976, pp. 181–186).

Frege, G. (1897), 'Über die Begriffsschrift des Herrn Peano und meine eigene', *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig: Mathematisch-physische Klasse* **48**, 361–378. English translation by V. H. Dudman in (Frege 1984, pp. 234–248).

Frege, G. (1903), *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*, Vol. II, Hermann Pohle, Jena. English translation by P. Ebert and M. Rossberg in (Frege 2013).

Frege, G. (1964), *Begriffsschrift und andere Aufsätze*, Georg Olms, Hildesheim. English translation by T. W. Bynum in (Frege 1972).

Frege, G. (1972), *Conceptual Notation and Related Articles*, Clarendon Press, Oxford.

Frege, G. (1976), *Wissenschaftlicher Briefwechsel*, Felix Meiner, Hamburg.

Frege, G. (1984), *Collected Papers on Mathematics, Logic, and Philosophy*, Blackwell, Oxford.

Frege, G. (2013), *Basic Laws of Arithmetic*, Oxford University Press, Oxford.

Goldfarb, W. (1980), 'Review to H. C. Kennedy, G. Peano, *Selected Works of Giuseppe Peano*', *The Journal of Symbolic Logic* **45**, 177–180.

Grassmann, H. (1861), *Lehrbuch der Arithmetik für höhere Lehranstalten*, Adolph Enslin, Berlin.

Grattan-Guinness, I. (2000), *The Search for Mathematical Roots, 1870–1940*, Princeton University Press, Princeton.

Grattan-Guinness, I. (2011), Giuseppe Peano: a Revolutionary in Symbolic Logic?, *in* Skof (2011), pp. 135–141.

Haaparanta, L., ed. (2009), *The Development of Modern Logic*, Oxford University Press, Oxford.

Jourdain, P. E. B. (1914), Preface. In Couturat, L. (1914). *The Algebra of Logic*. English translation by L. G. Robinson. Chicago: Open Court, pp. iii–xiii.

Lolli, G. (2011), Peano and the Foundations of Arithmetic, *in* Skof (2011), pp. 47–66.

Peano, G. (1888), *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva*, Fratelli Bocca, Turin.

Peano, G. (1889), *Arithmetices principia nova methodo exposita*, Fratelli Bocca, Turin. English translation by H. C. Kennedy in (Peano 1973, pp. 101–134).

Peano, G. (1891*a*), 'Formole di Logica Matematica', *Rivista di matematica* **1**, 24–31.

Peano, G. (1891*b*), 'Principii di Logica Matematica', *Rivista di matematica* **1**, 1–10. English translation by H. C. Kennedy in (Peano 1973, pp. 153–161).

Peano, G. (1891*c*), 'Sul concetto di numero. Nota I', *Rivista di matematica* **1**, 87–102.

Peano, G. (1894), *Notations de logique mathématique* (*Introduction au Formulaire de mathématiques*), Guadagnini, Turin.

Peano, G. (1895*a*), *Formulaire de Mathématiques*, Vol. I, Fratelli Bocca, Turin.

Peano, G. (1895*b*), 'Recensione: Dr. Gottlob Frege, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet, Erster Band*, Jena, 1893', *Rivista di matematica* **5**, 122–128. English translation by V. Dudman in (Dudman 1971, pp. 27–31).

Peano, G. (1896–1897*b*), 'Studii di Logica Matematica', *Atti della Reale Accademia delle Scienze di Torino* **32**, 565–583. English translation by H. C. Kennedy in (Peano 1973, pp. 190–205).

Peano, G. (1897*a*), *Formulaire de mathématiques*, Vol. II, §1: Logique mathématique, Fratelli Bocca, Turin.

Peano, G. (1898*a*), *Formulaire de Mathématiques*, Vol. II, §2: Aritmetica, Fratelli Bocca, Turin.

Peano, G. (1898*b*), 'Risposta ad una lettera di G. Frege, preceduta dalla lettera di Frege', *Revue de Mathématiques* **6**, 60–61.

Peano, G. (1899), *Formulaire de Mathématiques*, Vol. II, §3: Logique mathématique. Arithmétique. Limites. Nombres complexes. Vecteurs. Dérivées. Intégrales, Bocca, Turin.

Peano, G. (1900), 'Formules de Logique Mathématique', *Revue de Mathématiques* **7**, 1–41.

Peano, G. (1973), *Selected works of Giuseppe Peano*, Allen & Unwin, London.

Peirce, C. S. (1880), 'On the Algebra of Logic', *American Journal of Mathematics* **3**, 15–57.

Schröder, E. (1890), *Vorlesungen über die Algebra der Logik* (*exakte Logic*), Vol. 1, G. Teubner, Leipzig.

Schröder, E. (1891), *Vorlesungen über die Algebra der Logik* (*exakte Logic*), Vol. 2 (1. Abteilung), G. Teubner, Leipzig.

Schröder, E. (1895), *Vorlesungen über die Algebra der Logik* (*exakte Logic*), Vol. 3, G. Teubner, Leipzig.

Skof, F., ed. (2011), *Giuseppe Peano between Mathematics and Logic*, Springer, Milano.

van Heijenoort, J. (1967*a*), 'Logic as Calculus and Logic as Language', *Synthese* **17**, 324–330.

van Heijenoort, J., ed. (1967*b*), *From Frege to Gödel, a Source Book in Mathematical Thought*, Harvard University Press, Cambridge.

von Plato, J. (2014), *Elements of Logical Reasoning*, Cambridge University Press, Cambridge.

von Plato, J. (2017), *The Great Formal Machinery Works: Theories of Deduction and Computation at the Origins of the Digital Age*, Princeton University Press, Princeton.

von Plato, J. (2018), The Development of Proof Theory. *The Standford Encyclopedia of Philosophy* (Winter 2018 Edition), Zalta, E. (Ed.). `https://plato.stanford.edu/archives/win2018/entries/proof-theory-development/`.

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