FREGE, PEANO AND THE INTERPLAY BETWEEN LOGIC AND MATHEMATICS

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ABSTRACT. [*English*] In contemporary historical studies, Peano is usually included in the logical tradition pioneered by Frege. In this paper, I shall first demonstrate that Frege and Peano independently developed a similar way of using logic for the rigorous expression and proof of mathematical laws. However, I shall then claim that Peano also used his mathematical logic in such a way that anticipated a formalisation of mathematical theories which was incompatible with Frege's conception of logic.

[*French*] Dans les études historiques contemporaines, les contributions de Peano sont généralement envisagées dans le cadre de la tradition logique initiée par Frege. Dans cet article, je vais d'abord démontrer que Frege et Peano ont développé de manière indépendante des approches semblables visant à s'appuyer sur la logique pour exprimer rigoureusement des lois mathématiques et les prouver. Ensuite, je soutiendrai cependant que Peano a également utilisé sa logique mathématique d'une manière qui anticipait la formalisation des théories mathématiques, laquelle est incompatible avec la conception de la logique défendue par Frege.

1. INTRODUCTION

Even by the early twentieth-century, in Jourdain's Preface to the English translation of Couturat's *L'Algèbre de la Logique* [Jourdain, 1914, viii], Frege and Peano had been presented as members of the same logical tradition. The alleged proximity of the views of Frege and Peano, purportedly synthesised by Russell, has been retained in the contemporary historiography of logic and has become a commonplace.[1] Frege's and Peano's conceptions of logic stand out in opposition to the algebra of logic tradition.

In this paper I shall question Peano's inclusion in the Frege-Russell tradition on the basis that Peano developed a specific application of logic to mathematics that was incompatible with Frege's view. First, I shall argue that Frege intended to use the logical system developed in his mature works not only to show that arithmetic could be reduced to logic but also as a tool for the rigorous expression and proof of mathematical laws. Second, I shall propose that although Peano devised a reformulation of mathematical theories by means of logic similar to Frege's, in addition, Peano – and the members of the so-called Peano school – developed a new understanding of

[1] Van Heijenoort [1967*b*] develops Jourdain's dichotomy of two logical traditions in terms of the "logic as language" tradition and the "logic as calculus" tradition. This paper became very influential and established a conceptual framework for the history of modern logic.

the resulting expressions of this reformulation that anticipated a contemporary notion of formalisation which Frege could not accept.[2] In sum, I shall investigate Frege's and Peano's views on the application of logic to mathematical theories and the formalisation of the latter, and conclude that they developed accounts that were, in significant respects, irreconcilable.

This paper is in two parts. First, I shall discuss Frege's views on the application of logic to arithmetic. This involves his logicist project but also, and crucially, his proposal to apply the formal resources of logic to a reformulation of mathematical theories. Second, I shall study, on the one hand, Peano's aim of creating an ideography by means of the combination of logical and mathematical symbols and, on the other, the development by the members of Peano's school of a new understanding of the expressions of such an ideography in the context of proofs of independence.

2. Frege's reduction and symbolisation

2.1. For a significant stretch of his career, Frege understood the relationship between arithmetic and logic as the *reduction* of the former to the latter. Frege's logicist project consists in the proof that arithmetic is a logical theory. In *Grundlagen der Arithmetik* [1884], Frege considers the logicist project from a philosophical point of view and tries to informally justify that the reduction can be carried out. He then attempts a formal proof of the reduction of arithmetic to logic in *Grundgesetze der Arithmetik* [1893; 1903] (hereinafter, *Grundgesetze*).

One of the objectives of the logicist project is the explicit definition of the basic notions of arithmetic by means of the symbols of logic. This requires the development of a logical language with enough expressive power. In order to achieve this goal, in *Grundgesetze* Frege profoundly modifies the concept-script – the logical system he had first presented in *Begriffsschrift, eine der arithmetischen Formelsprache des reinen Denkens* [1879*b*] (hereinafter, *Begriffsschrift*). Among other things, in *Grundgesetze* he rigidly regiments quantification and incorporates the notion of value-range in the language by means of a function symbol, '$\dot{\epsilon}\varphi(\epsilon)$'.

Frege's logicist project also aims to prove that all arithmetical laws are logical laws, i.e., to prove that the laws of arithmetic can be derived in the calculus of the concept-script from logical laws and definitions. Such a proof involves a modification of the semantical status of some of the components of arithmetical laws; the letters occurring in them then go on to express generality over the domain of logical objects and, accordingly, the quantifiers cease to range exclusively over natural numbers. Therefore, after the reduction of arithmetic to the concept-script, arithmetical laws can still be interpreted judgements, although they would go on to express purely logical facts.

[2] In the context of this paper, I understand by '*formalisation*' the replacement of a set of sentences expressed in a language L (usually, natural language) with a corresponding set of sentences expressed in a *formal* language L' (typically, that of first-order logic), which preserves the logical form of the sentences of L and expresses it using logical symbols, but substitutes non-logical constants (which are uninterpreted) for the non-logical terms of L. On the notion of formal language, see [Church, 1956, 2–68].

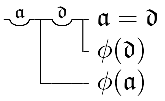
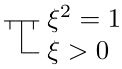
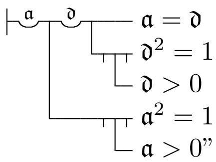
Moreover, after the explicit definition of the basic notions of arithmetic, for Frege there is no need to keep the symbols that represent them in the process of proving arithmetical laws by logical means. The proofs and judgements of the first volume of *Grundgesetze* do not contain arithmetical symbols, but the primitive symbols of the concept-script, letters and symbols that Frege introduces by means of definitions, such as '0' and '1' – which refer to the cardinal numbers 0 and 1, respectively.

All in all, the reduction of a theory to another is significantly different from the formalisation of a theory. A formalisation requires a formal language or, at least, a symbolic language that contains non-logical constants. Since non-logical constants are uninterpreted, the resulting formulas of a formalisation do not preserve the meaning of the formalised sentences; only the syntactic status of the symbols of the formalised theory is kept. In contrast, a reduction does not require a formal language; in fact, it can only be performed by means of an interpreted language, since the original meaning of both the primitive symbols and the laws of the reduced theory have to be maintained in essence. In fact, the basic terms of the theory by means of which the reduction is performed are substantive, in the sense that they refer to the specific entities the theory is about.[3] This enables the provision of explicit definitions of the basic notions of the reduced theory and the preservation of their properties. For instance, in Frege's reduction of arithmetic to the concept-script, the cardinal numbers are defined as logical objects, but at the same time they retain their mathematical properties.

2.2. As is well known, Frege's logicist project ended abruptly with the discovery of the inconsistency of the formal system presented in *Grundgesetze*. After 1902, Frege was forced to modify his views on the relation between logic and arithmetic. The best witness to Frege's post-logicist understanding of the relationship between the concept-script and mathematics can be found in the student notes Carnap wrote while attending some of Frege's courses in Jena between 1910 and 1913 [Frege, 1996]. In the first of these courses, *Begriffsschrift I* (which took place in the winter semester of 1910–1911), Frege presents the main components of the language of the concept-script – as they are described in *Grundgesetze*, but without mentioning the symbols for value-ranges or for the function $\backslash\xi$. He thus obtains a higher-order logical language. Frege then shows, with examples, how its syntax could be naturally adapted to the expressions of arithmetic. This process consists in connecting atomic expressions of number theory, such as '$a > 0$' or '$(a - b) + b = a$', using the logical symbols of the concept-script. The combination of atomic expressions of number theory and logical symbols also involves the incorporation of the letters of the concept-script – by means of which generality is expressed – into the aforementioned atomic expressions. For instance:[4]

[3] I take the notion of substantive basic terms and their role in the reduction of a mathematical theory from [Klev, 2011].

[4] When an English translation is quoted, two page numbers – separated with a semicolon – are given: the first corresponds to the most recent edition of the source listed and the second to the English translation. When no English translation is available, quotes and page numbers are taken from the most recent edition of the source and translated by the author.

> "If we want to express that at most *one* object falls under a concept, we write:
>
>  e.g., [the concept] positive square root of 1: 
>
> 
>
> [Frege, 1996, 17; 77]

In the second course, *Begriffsschrift II* (which took place in the summer semester of 1913), Frege first presents the logical fragment of the calculus of the *Grundgesetze* concept-script: he introduces the basic laws and some of its inference rules, but omits basic laws (V) and (VI), which involve value-ranges.[5] Frege then exemplifies how the calculus of the concept-script could be applied to prove two theorems of analysis. These proofs are detailed reconstructions of mathematical proofs using the formal tools provided by the concept-script. First, Frege reformulates the theorem he wants to justify using a combination of logical and mathematical symbols. Second, he lists and reformulates in the explained way the propositions of analysis that are needed in the proof as premises. Third, the logical principles that are required in the proof are incorporated as premises by means of substitutions, in such a way that atomic expressions belonging to the language of the concept-script, such as '$M_\beta(f(\beta))$' or '$f(a)$' (which, strictly speaking, should be considered terms) are replaced with expressions of analysis. With all these components, Frege conducts the proof in a similar way as he had done in *Grundgesetze*: he renders explicit all the logical principles and formal steps involved, using the inference rules available.

Frege's methodology and goals in these courses coincides with the application of the concept-script he devised during the immediate years after the publication of *Begriffsschrift*, in the papers 'Anwendungen der Begriffsschrift' [1879*a*], 'Booles rechnende Logik und die Begriffsschrift' [1880–1881] and 'Über den Zweck der Begriffsschrift' [1882]. In these papers he is explicit about the aim of such a combination of the concept-script with a scientific theory: Frege strongly associates it with the rigorous expression of the laws and proofs of such a theory. He rejects the perspective of producing what he calls an 'abstract logic', i.e. a symbolism isolated from the expression of specific meaning. As he says in 'Über den Zweck der Begriffsschrift', in which he compares the 1879 concept-script with Boolean logic, "I did not wish to present an abstract logic in formulas, but to express a content through written symbols in a more precise and perspicuous way than is possible with

[5]There is no mention of basic law (IV) in the student notes. However, this basic law belongs to the propositional fragment of the concept-script and is completely unrelated to the notion of value-range. In the notes, right before basic law (III) is introduced by Frege, several pages are empty – which indicates that Carnap missed some lectures or failed to take notes in them. Either Frege mentioned basic law (IV) during the course and Carnap did not record it or Frege considered that this basic law was unnecessary for his purposes in this course and did not mention it.

words" [1882, 97; 90–91]. Also in this paper Frege offers a general overview of how he intends to apply his concept-script to arithmetic:

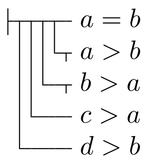
> "Now I have attempted to supplement the formula language of arithmetic with symbols for the logical relations in order to produce – at first just for arithmetic – a concept-script[6] of the kind I have presented as desirable. This does not rule out the application of my symbols to other fields. The logical relations occur everywhere, and the symbols for particular contents can be so chosen that they fit the framework of the concept-script." [Frege, 1882, 113–114; 89]

Frege's view in this passage coincides with the use of the concept-script described in the 1910–1913 courses – that of a formal structure that could be combined with the atomic expressions of mathematical theories in such a way that the meaning of the laws of these theories could be expressed in a precise way and their proofs could be conducted with the standards of rigour of the concept-script.[7]

2.3. The application of the concept-script which Frege proposes both in his 1879–1882 papers and in the post-*Grundgesetze* courses departed from a formalisation. For the sake of clarity, I shall refer to Frege's proposed application of the concept-script to a scientific theory as '*symbolisation*'.

Frege wants to preserve the symbols of arithmetic and use them as canonical names, i.e., as symbols with a specific and fixed meaning. Even quantification is restricted in this application; in the examples in the student notes, all letters are supposed to range over real numbers, since the numerical operations and relations are only defined for them:

> "And we use:



> This [$c > a$ and $d > b$] is supposed to mean that a and b are real numbers, since it is only for them that $>$ is supposed to be defined."
> [Frege, 1996, 26; 101]

Note that Frege shows no difficulty in restricting the domain of the letters or the applicability of arithmetical relations. In this context, if the letters 'a' and 'c' were to have a domain wider than the set of the real numbers, then it would not be possible to determine the meaning of an expression such as '$c > a$', since the relation $>$ is, as Frege acknowledges, only defined for real numbers as arguments.

[6]For the sake of terminological homogeneity, I have replaced 'conceptual notation' with 'concept-script' as the English counterpart of the German '*Begriffsschrift*' in this quote, taken from Bynum's translation of [Frege, 1882].

[7]Frege's position regarding the application of the concept-script to the rigorization of mathematical theories is related to his project of creating a *lingua characterica*. This latter notion can be connected to Leibniz's ideal of a scientific language. The choice of the term 'concept-script' (*Begriffsschrift*) is also related to this project. On Frege's notion of *lingua characterica* and its relation to the concept-script, see [Bertran-San Millán, 2020*a*]. See also [Patzig, 1969], [Kluge, 1977], [Peckhaus, 2004] and [Korte, 2010].

The symbols of number theory used in this application are not employed by Frege to express abstract properties and relations. They are not, therefore, seen as uninterpreted non-logical constants devoid of meaning. Only the letters have a specific domain, determined by the intended application.

By means of a symbolisation, Frege aims to overcome the lack of precision posed by the use of natural language in the definition of the derived concepts of mathematical theories and in their proofs. These theories do not have the expressive means necessary for the symbolic representation of the logical relations that form complex sentences. At the same time, most derived notions are defined in complex sentences. As a consequence, the derived notions, if defined at all, have to be defined using natural language, by means of which it is not possible to attain the level of exactness and rigour Frege required for mathematics. Likewise, in 'Booles rechnende Logik un die Begriffsschrift', while comparing his concept-script with Boolean logic, Frege states the following:

> "[The concept-script] is in a position to represent the formation of the concepts actually needed in science, in contrast to the relatively sterile multiplicative and additive combinations we find in Boole." [Frege, 1880–1881, 52; 46]

From 1910–1913 Frege's remark on the poor expressive capabilities of Boolean logic had to be qualified. By then, the proponents of the algebra of logic had overcome all the expressive shortcomings they had faced in 1880. However, he retained unmodified his claim that the language of mathematical theories needed to be complemented with the formal resources of the concept-script if their new concepts were to be defined with an adequate standard of rigour. In these courses, Frege even appeals to the same examples as those of 'Booles rechnende Logik un die Begriffsschrift' [1880–1881] – namely, the definition of the notion of the continuity of a function – in order to show the fruitfulness of his symbolisation in the processes of concept formation.

In fact, by applying the formal resources of the concept-script to a scientific theory such as analysis, Frege shows a lack of interest in logic as a subject matter. He takes great pains to carefully show how a proof of a theorem of analysis can be performed using the formal resources of the concept-script,[8] but there is absolutely no evidence in the 1879–1883 papers or in the 1910–1913 courses to show that by symbolising analysis or arithmetic in the way he does he intends to answer metatheoretical questions such as the completeness or consistency of these theories, or the independence of their axioms. The focus is put on precision and rigour. In 'Booles rechnende Logik und die Begriffsschrift', after a full symbolisation of the proof of an arithmetical theorem [1880–1881, 30–36; 27–32] – analogous to those performed in *Begriffsschrift II* [1996, 25–37; 98–119] – Frege lists the demands fulfilled by such a symbolisation: a complete and clear specification of all the principles necessary for the derivation of the theorem; a warrant that the proof contains no appeal to intuition; and, finally, the certainty that there are no formal steps missing in the proof, since all of them have been rendered explicit [1880–1881, 36; 32].

[8]More than a third of the pages that correspond to Carnap's notes on *Begriffsschrift II* are devoted to the proof of a single theorem. See [Frege, 1996, 29–37; 103–119].

3. Peano's symbolisation and formalisation

3.1. One of the most prominent elements in Peano's development of his logic, which he calls 'mathematical logic', is the construction of a logical symbolism that can be used as a tool for the rigorous expression of the laws of scientific theories as well as for helping making explicit the logical principles involved in their proofs. Even in Peano's first uses of his logical symbolism, rigour in the derivation of theorems and a precise characterisation of scientific terms are already established as the main goals of this reformulation of scientific theories. In his seminal *Arithmetices principia nova methodo exposita* [1889*a*] (hereinafter, *Arithmetices principia*) Peano expressed himself thus:[9]

> "With this notation every proposition assumes the form and precision equations enjoy in algebra, and from propositions so written others may be deduced, by a process which resembles the solution of algebraic equations. That is the chief reason for writing this paper.
>
> [...] Those arithmetical signs which may be expressed using others along with signs of logic represent the ideas we can define. Thus I have defined every sign, if you except the four which are contained in the explanations of §1." [Peano, 1889*a*, 21; 102]

As we shall see, Peano's intended use of mathematical logic can be seen as a symbolisation. In this sense, he shares Frege's view on the combination of logic and scientific theories for the construction of a symbolised reformulation of these theories.[10]

Still, Peano's understanding of a symbolisation cannot be reduced to a mere rewriting of sentences in the natural language by means of which the laws of scientific theories are expressed. He produces what he called 'ideography' and thus connects it – just as Frege had done via Trendelenburg's [1856] use of the term '*Begriffsschrift*' – with Leibniz's scientific ideal of a *characteristica universalis*, which is not intended to express uttered sounds but to represent the structure of concepts.[11]

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[9]In *Principii di Geometria logicamente esposti* [1889*b*] (hereinafter, *Principii di Geometria*) Peano discusses several axioms of Pasch's axiomatisation of geometry – which are formulated in natural language – and contrasts them with his own axioms – which are symbolised [1889*b*, 84–85]. In this discussion he highlights the ambiguities involved in the expression of mathematical laws by means of natural language.

[10]Since Frege first developed his approach to the symbolisation of mathematical theories as early as 1879, it might be asked whether he influenced Peano's notion of symbolisation using mathematical logic. It is very unlikely. Peano's first symbolisation was presented in *Arithmetices principia*, published in 1889, while – as Nidditch [1963, 105] states – he first refers to Frege in [1891*b*, 101, note 5; 155, fn. 5]. Prior to 1891, Peano had published other papers in which he symbolised mathematical theories: [1889*b*], [1890*b*] and [1890*a*]. More importantly, Frege articulates in full the application of the concept-script to logic in [1880–1881]; however, he attempted three times but failed to publish this paper, so probably Peano never had access to it. The extant correspondence between Frege and Peano started in 1891 – no trace of any previous letter can be found – and there Frege neither mentions any of his 1879–1882 papers nor explains how he conceives the application of the concept-script to mathematical theories.

[11]As Barnes [2002] argues, the term '*Begriffsschrift*' can be translated as 'ideography'. On the relation between Peano and Leibniz's scientific ideal, see [Cantù, 2014].

In *Notations de logique mathématique*, Peano reflects on the creation of an ideography. He describes a similar process to Frege's symbolisation. As a first step, Peano proposes to extract the logical form of the sentences of a given theory and to express it using the symbols of logic. He then suggests an analysis of the terms of the theory, by means of which its primitive terms can be located and their connection with the other terms can be discovered. This last step makes it clear that Peano does not intend to perform a mere rewriting of the theory.[12] In his words:

> "Any theory can be reduced to symbols, for every spoken language, and every writing, is a symbolism, or a series of signs that represent ideas. In order to apply the signs we have explained, we can take the propositions of the theory in question, written in ordinary language, and replace the word *is* with the signs ε, $=$, $\supset$, as the case may be, and [put] instead of *and*, *or*, ... the signs $\cap$, $\cup$, ...; and that *cum granu salis*, because we saw for instance that, depending on the position, the conjunction *and* is represented by means of $\cap$ or $\cup$.
>
> After this first transformation, the propositions are expressed in a few words, linked by the logical signs $\cap$, $\cup$, $=$, $\supset$, etc.; and if it has been well done, the words that remain are devoid of any grammatical form; for all the relations of grammar are expressed by means of the signs of logic. These words represent the proper ideas of the theory being studied. Then the ideas represented by these words are analysed, the composed ideas are decomposed into the simple parts, and only, after a long series of reductions and transformations, one obtains a small group of words, which can be considered as minimum, by means of which, combined with the signs of logic, all the ideas and propositions of the science under study can be expressed." [Peano, 1894*a*, 164]

With this ideography, i.e., with the combination of mathematical logic and the primitive terms of the language of scientific theories, Peano can eliminate all trace of natural language in the formulation of these theories. Since their primitive terms are preserved, the original meaning of the expressions of these theories is also kept.

Peano refers to his symbolisation of scientific theories as a reduction. However, he does not intend to define the primitive notions of a theory in terms of another (in this case, mathematical logic), nor prove that the axioms of the former are, in fact, theorems of the latter. In this sense, Peano's notion of reduction does not correspond to the characterisation of Frege's reduction of arithmetic to the concept-script provided in Section 2.1.[13]

[12]See also [1896–1897, 203; 191], where Peano distinguishes between a symbolisation – by means of an ideography – and a mere rewriting.

[13]Neither did Frege see Peano's symbolisation as a reduction. See [Frege, 1897, 365–366; 237]. Although there has been a debate in the literature, some consensus has arisen over the thesis that Peano did not endorse Frege's logicist project. Most historical studies plainly deny that Peano was a logicist (see [Kennedy, 1963, 264], [Segre, 1995], [Lolli, 2011]), while others also emphasise his rejection of philosophical discussions (see [Geymonat, 1955]). See also [Grattan-Guinness, 2000, 247–249].

Peano focusses on the ideographic reformulations of mathematical theories. With the axiomatic method in mind, he produces several symbolic axiomatisations of arithmetic and geometry.[14] The resulting theories are constituted by two separate groups of axioms: a set of logical principles (which usually includes principles of the logic of classes) and a set of mathematical axioms.[15] The clear separation of the logical and the mathematical constituents of the theory is shared with Frege.

Moreover, Peano's presentation of the language of a symbolised mathematical theory also preserves this distinction. Peano consistently provides specific lists for logical (and class-theoretical) and mathematical symbols and treats the latter as substantive, as canonical names.[16] In this regard, later in 1897 he expresses the convenience of preserving the symbols of arithmetic:

> "The symbols of Algebra allow us to express some propositions:
>
> $$2 + 3 = 5, \qquad 5 < 7, \ldots$$
>
> We keep these symbols; sometimes we even generalise their meaning; but when we encounter ideas that cannot be expressed by the symbols of Algebra, we introduce new symbols. For instance, we want to express the proposition
>
> 7 is a prime number;
>
> we already have a symbol to indicate the subject 7; we introduce a symbol Np to signify 'prime number'; and a symbol ε to indicate 'is a'; then the stated proposition is transformed into
>
> $$7 \,\varepsilon\, \mathrm{Np}."$$
>
> [Peano, 1897, 241]

Peano is aware that the expression of mathematical principles and the definition of derived notions requires, besides the use of logical symbols and symbols of the calculus of classes, the enlargement of the set of primitive symbols. For instance, he introduces 'N' and 'Np' to refer to the class of natural numbers and the class of prime numbers, respectively. These new symbols should also be taken as canonical names, although they do not belong to the language of arithmetic *sensu stricto*. After all, Peano includes them in the list of primitive symbols of his symbolisation of arithmetic and

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[14] Having presented the first formulation of his mathematical logic in *Arithmetices principia*, during the early 1890s Peano provides multiple examples of the ideographic formulations of mathematical theories: analysis (see, for instance, [1890*a*; 1892*a*]), geometry (see [1889*b*; 1894*b*]), arithmetic (see, for instance, [1891*d*]) and even Euclid's *Elements* (see [1890*b*; 1891*c*; 1892*b*]). Peano's major ideographic endeavour is his collective project of a *Formulaire de mathématiques*, which was published in several volumes and revised in subsequent editions. On this project, see [Borga, Freguglia *et al.*, 1985, 163–170] and [Roero, 2011].

[15] This claim should be qualified if the earliest formulations of Peano's mathematical logic are considered. Peano does not axiomatise the logical component of his axiomatisation of arithmetic presented in *Arithmetices principia*. He first offers an axiomatic presentation of the calculus of propositions in [1891*a*]. Moreover, in *Principii di Geometria* the logical principles are not explicit. In this work Peano only includes three axioms that involve equality [1889*b*, 61].

[16] See, for instance, his presentation in *Arithmetices principia* [1889*a*, 23; 103–104]. On Peano's view regarding the substantivity of the primitive notions of geometry and arithmetic, see [Borga, Freguglia *et al.*, 1985, 51–54, 88–94, 109–110].

assumes that they express basic properties of numbers that have been left undefined.

By the end of the nineteenth century, geometry lacked a symbolic language like that of arithmetic. In this sense, Peano's symbolisation of geometry could not preserve established geometrical symbols that were already in use: the list of primitive symbols of geometry had to be created anew. In *Principii di Geometria*, adopting '1' and 'ε' as primitive symbols, and using the symbols of the language of his mathematical logic introduced in *Arithmetices principia*, he offers a symbolisation of geometry and presents the theory axiomatically. Peano thus employs a mixture of logical symbols, arithmetical symbols and symbols of the calculus of classes and assigns the latter two a geometrical meaning (sometimes preserving, for certain applications, their original meaning). For instance, '1' is used to refer to the class of points and 'ε' to the relation between a point and a segment [1889*b*, 59–61]. However, at the same time, Peano would express that the objects a and b are points by '$a, b \,\varepsilon\, 1$', where 'ε' is used as the symbol for membership.

3.2. Almost simultaneously to his work on the symbolisation of mathematical theories, Peano developed a new understanding of symbolised expressions that was intimately connected with the evaluation of the independence of the axioms of these theories. As we shall see below, this new understanding of symbolised mathematical axioms in the context of proofs of independence anticipates in significant ways a formalisation.

Peano does not explain in detail the nature of this new understanding of symbolised mathematical laws. However, some members of the so-called Peano school offer lengthy accounts that are related to their explanation of the resolution of metamathematical questions such as the independence of the axioms or the primitive notions of mathematical theories. These accounts can shed light on Peano's position.

A fundamental element for the understanding of a symbolised mathematical theory in Peano's school is the stratification of the components of this theory. In *Arithmetices principia*, Peano distinguishes between the axioms and the theorems of arithmetic, and also between its defined and undefined symbols:

> "Those arithmetical signs which may be expressed by using others along with signs of logic represent the ideas that we can define. Thus I have defined every sign, if you except the four which are contained in the explanations of §1 [N, 1, +1, =]. If, as I believe, these cannot be reduced further, then the ideas expressed by them may not be defined by ideas already supposed to be known.
>
> Propositions which are deduced from others by the operations of logic are *theorems*; those for which this is not true I have called *axioms*. There are nine *axioms* here (§1), and they express fundamental properties of the undefined signs." [Peano, 1889*a*, 21; 102]

In 'Formole di Logica Matematica' [1891*a*, 102–104] Peano rephrases this double distinction in terms of primitive and derived propositions and symbols. Primitive propositions, or axioms, are left unproved and primitive symbols are not defined. By means of definitions in terms of primitive symbols all derived symbols can be obtained, and theorems (i.e., derived propositions) are the result of derivations that start from primitive propositions and definitions. This idea refines Peano's view on the process of the creation of an ideography.

From this conceptual framework, Padoa characterises in 'Essai d'une théorie algébraique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque' [1901] what he calls a 'deductive theory'. The components of a deductive theory are expressed in a language constituted by a system of primitive symbols (which Padoa calls 'undefined symbols'), while the theory is determined by a system of primitive propositions ('unproved propositions' in Padoa's terminology). The deductive approach is defined by the disentanglement of these systems of symbols and propositions from their original meaning:

> "[D]uring the period of *elaboration* of any deductive theory we choose the *ideas* to be represented by the undefined symbols and the *facts* to be stated by the unproved propositions; but, when we begin to *formulate* the theory, we can imagine that the undefined symbols are *completely devoid of meaning* and that the unproved propositions (instead of stating *facts*, that is, *relations* between the *ideas* represented by the undefined symbols) are simply *conditions* imposed upon undefined symbols.
>
> Then, the *system* of *ideas* that we have initially chosen is simply *one interpretation* of the *system* of *undefined symbols*; but from the deductive point of view this interpretation can be ignored by the reader, who is free to replace it in his mind by *another interpretation* that satisfies the conditions stated by the *unproved propositions*. And since these propositions, from the deductive point of view, do not state *facts*, but *conditions*, we cannot consider them true *postulates*." [Padoa, 1901, 318; 120-121]

This way of understanding a theory is thus not meant to preserve its content and express it in a rigorous way, as is the case in the symbolisation that Peano himself or Frege developed. On the one hand, primitive symbols are detached from their original meaning and are effectively seen as non-logical constants, that is, as uninterpreted symbols, whereas on the other hand, primitive propositions cease to be seen as expressing true facts; they express the conditions that an interpretation must hold in order to satisfy them.[17]

This process entails that the propositions of the formalised theory only express abstract relations between unspecified objects, properties and relations. In this sense, the development of a formalised theory (i.e., in Padoa's terminology, a deductive theory) involves only a deductive relation between primitive and derived propositions:

> "[F]or what is necessary to the logical development of a deductive theory is not *the empirical knowledge of properties of things*, but *the formal knowledge of relations between symbols*." [Padoa, 1901, 319; 121]

[17]The fact that Peano and Padoa talk about interpretations and considered specific domains and interpretations for non-logical symbols does not mean that they anticipate the contemporary notion of model. On the differences between the notion of interpretation common to Peano and Padoa as opposed to the contemporary notion of model see [Mancosu, Zach *et al.*, 2009, 323–324].

The distinction between primitive and derived symbols, and between axioms and theorems, guarantees that by merely providing an interpretation of the primitive symbols that satisfies the axioms, the whole theory is satisfied. All relations between primitive and derived symbols are made explicit through definitions and, similarly, all theorems are deduced from axioms and definitions.[18]

Each of Peano, Padoa and Pieri insist upon putting the notion of deduction at the centre of their accounts of the formalisation of mathematical theories. However, they never characterise precisely this notion. In their works, deduction remains an informal notion that is not formally defined. Peano does offer several specifications of logical principles in his presentations of the mathematical logic, but all things considered he fails to provide a full characterisation of the notion of deduction: crucially, a complete system of inference rules cannot be found in Peano's presentations of mathematical logic.[19]

The hierarchic structure of a mathematical theory proposed by the members of Peano's school also involves some methodological principles that would determine their work on metamathematical questions. Since a deductive theory is built from a system of primitive propositions and a system of primitive symbols, the independence of these propositions and the irreducibility of these symbols is understood as a methodological goal. As Pieri puts in in 'Sur la Géométrie envisagée comme un système purement logique':

> "As far as possible, primitive ideas should be *irreducible* to one another, so that none of them can be explicitly defined by means of others; and, similarly, the postulates should be *independent* of each other, so that none can be deduced from the others." [Pieri, 1901, 380]

It is thus understandable that, right after providing an axiomatisation of a mathematical theory, Peano studies the independence of their axioms.[20]

4. Concluding remarks

In this paper I have focused upon the views of Frege and Peano on the application of logical symbolism and the methods of logic to mathematical theories, and concluded that they disagreed as regards substantial aspects.

[18]As Blanchette [2017, 47] states, the stratification of a mathematical language in terms of primitive and derived symbols is instrumental to the understanding of reinterpretation as a method for proving independence, and can be seen as a distinctive feature of late nineteenth-century approaches to the independence of the axioms of geometry. The pioneering work of Peano's school in this regard should not be underestimated, especially because of the fact that it predates by a decade Hilbert's work on this field.

[19]Peano's metatheoretical questions are, to a great extent, intuitively answered. Although he does not consider the notion of soundness, his results in metamathematics presuppose that the calculi he used are sound. For a discussion on the claim that Peano does not adopt a fully deductive approach to logic, see [Bertran-San Millán, 2020*b*]. See also [Goldfarb, 1980]. For a critical approach to this claim, see [von Plato, 2017, 50–57].

[20]Peano's independence arguments in geometry can be found in [1889*b*; 1894*b*]. Most of Peano's proofs of independence have the axioms of arithmetic as their object. See [Peano, 1889*a*, 1891*d*, 1897, 1898, 1899, 1901].

Their varying views on the formalisation of mathematical theories are rooted in a deep disagreement regarding their goals. For a significant part of his career, Frege aimed at showing that arithmetic could be reduced to logic. Before this project was fully articulated and after it had failed, he intended to use logic as a formal structure appropriate to supplement the language of arithmetic. Peano never attempted to reduce arithmetic to logic, but he also devised – independently of Frege – a symbolisation of mathematical theories with the assistance of logic. However, Peano also aimed at answering metatheoretical questions such as the independence of the axioms of a mathematical theory, and he developed an alternative understanding of symbolised expressions to fulfil this aim.

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