PEANO'S GEOMETRY: FROM EMPIRICAL FOUNDATIONS TO ABSTRACT DEVELOPMENT

JOAN BERTRAN-SAN-MILLÁN

Abstract. In *Principii di Geometria* [1889b] and 'Sui fondamenti della Geometria' [1894] Peano offers axiomatic presentations of projective geometry. There seems to be a tension between two poles in Peano's account: on the one hand, the view that the basic components of geometry must be founded on intuition, and, on the other, Peano's advocacy of the axiomatic method and an abstract understanding of the axioms.

By studying Peano's empiricist remarks and his conception of the notion of mathematical proof, and by discussing his critique of Segre's foundation of hyperspace geometry, I will argue that these two poles can be understood as compatible stages of a single process of construction rather than conflicting options.

Keywords: Peano · geometry · axiomatic method · empiricism · deductivism.

1. Introduction

During the last decades of the nineteenth century, foundational studies became a major field in geometrical research. In Italy, the publication of Fano's translation into Italian [1889]—made at Segre's request—of Klein's *Vergleichende Betrachtungen über neuere geometrische Forschungen* [1872] (commonly known as the *Erlangen program*) bolstered foundational investigations. Pasch's *Vorlesungen über neuere Geometrie* [1882] is also a key point of reference in this regard.

The growing importance of foundational studies ran parallel to the central role algebraic and projective geometry acquired in the last half of the century. The analytic development of projective geometry pioneered by geometers such as Plücker and Cremona made a pronounced impact on Italian scholarship.[1] Grassmann's groundbreaking *Ausdehnungslehre* [1844; 1862] attracted attention in Italy from the late 1850s.[2] Klein's two papers on non-Euclidean geometry [1871; 1873] also played an important role. Furthermore, the effort at providing coordinates for projective geometry exclusively on a geometrical basis led by von Staudt was followed by De Paolis in 'Sui fondamenti della geometria proiettiva' [1880–81].[3]

All in all, the intense developments to which geometry was subjected in the second half of the nineteenth century became the fertile ground from

Date: June 30, 2022 (**v1.1**).

[1]See [Plücker, 1828–31], [Plücker, 1868] and [Cremona, 1873].

[2]On the Italian reception of Grassmann *Ausdehnungslehre*, see [Bottazzini, 1985, pp. 27–34].

[3]Von Staudt's *Geometrie der Lage* [1847] was translated into Italian by Pieri [1889], again at the request of Segre.

which the Italian school of algebraic geometry and Peano's school could blossom, gaining international renown.[4]

Peano's work on geometry can be divided according into two main areas: the development of a geometrical calculus and the axiomatization of elementary and projective geometry from a synthetic point of view. In this paper, I will focus on this second aspect.[5] Specifically, I will investigate Peano's axiomatizations in *I Principii di Geometria logicamente esposti* [1889b] (hereinafter, *Principii di Geometria*) and 'Sui fondamenti della Geometria' [1894].

There seems to be a tension in Peano's construction of geometry in these two works. On the one hand, Peano insists that the basic geometrical concepts and propositions must have an empirical foundation. On the other hand, geometry starts from axioms, which cannot be attached to a single interpretation. In fact, Peano highlights the abstract character of the terms occurring in such axioms and argues that the demonstration of theorems from these axioms must proceed exclusively by logical means.[6]

By studying Peano's axiomatization of geometry, I will argue that the tension can be dissolved if these two seemingly contradictory positions are understood as compatible aspects of a single process of construction, rather than competing options. Specifically, I will explain that each stance corresponds to a specific phase in the construction of geometry. I will describe these two phases, and charaterize their relationship by referring to a dispute between Peano and Segre. Accordingly, I will first claim that for Peano, the construction of geometry must rely on a pre-mathematical phase determined by the selection of a minimal set of axioms and fundamental concepts, which have to be verifiable by direct observation. Second, I will argue that the formulation of the axioms entails a selection, rearrangement and systematization of content given intuitively. I will claim that, although there is a close connection between the content of the axioms and the nature of the fundamental notions of geometry, the former do not completely determine the latter. In Peano's construction of geometry, there is a second phase, properly mathematical, where rather than being attached to a single system of objects as their sole interpretation, the axioms are understood as abstract postulates.

A study of Peano's criticism of Segre's treatment of hyperspace geometry will allow me to substantiate Peano's abstract understanding of the axioms. On the one hand, Peano's opposition to a purely abstract construction of geometry is motivated by its lack of empirical foundation, and hence relies on his requirements regarding the first pre-mathematical phase. On the other

[4]For a panoramic view of nineteenth-century geometry, see [Gray, 2007]. On the connection between the development of projective geometry and modern logic, see [Eder, 2021]. On the development of projective geometry in Italy, see [Avellone *et al*, 2002].

[5]On Peano's geometrical calculus, see [Bottazzini, 1985] and [Borga *et al*, 1985, pp. 177–198]. On the relationship between Peano's geometrical calculus and the axiomatization of geometry, see [Gandon, 2006] and [Rizza, 2009].

[6]Although some historical studies emphasize the abstract aspect in Peano's construction of geometry (see [Kennedy, 1972]), others have observed the tension between empiricism and an abstract approach (see [Bottazzini, 2001, pp. 288–290], [Avellone *et al*, 2002, pp. 378–386], [Gandon, 2006, p. 253]). [Rizza, 2009] also aims at dissolving this apparent tension.

hand, Peano's abstract conception of postulates, in the second phase, can be better understood by alluding to two related notions of purity of method. Peano's advocacy of synthetic geometry, and thus for the independence of this discipline from metric considerations, is closely connected with his conception of the relation between the means of proof of theorems and their content. In Peano's view, the content of geometrical laws is not determined by their informal wording, but rather by the deductive relations they establish with the axioms. This indicates that, in the properly mathematical phase, the specific meaning conveyed by these laws becomes irrelevant. From this stance, I will argue that Peano's abstract axiomatic approach can be framed within deductivism. In fact, deductivism squares in a natural way with Peano's notions of purity and his understanding of mathematical proofs regimented by logical means.

This paper is organized into three parts. In the second section I will characterize Peano's understanding of the basic concepts of geometry and the requirement that they be empirically-founded. In the third section, I will explore Peano's critique of Segre's hyperspace geometry in order to contrast the former's empiricist stance with the latter's purely abstract approach. I will also describe Peano's conception of the content of geometrical propositions, and give his view on the nature of Desargues's theorem. This conception of content will inform, in the fourth section, Peano's views on the process of demonstration of geometrical propositions. From this standpoint, I will offer an explanation of Peano's abstract understanding of the axioms.

2. Empirical foundation of geometry

Peano's conception of the construction of a mathematical theory relies on a distinction between undefined and derived notions, and between unproven propositions, namely axioms or postulates, and theorems. In 'Sui fondamenti della Geometria' the undefined notions, the most basic concepts of geometry, are called '*primitive* notions' [1894, p. 116].[7] Peano states that the primitive notions must be "very simple ideas, common to all men" and "reduced to a minimum number" [1894, p. 116].

Both in *Principii di Geometria* [1889b, p. 77] and in 'Sui fondamenti della Geometria' [1894, p. 119], Peano states that the concepts of point and straight segment are the primitive notions of geometry. Specifically, the class of points **1** or p (as it is represented in *Principii di Geometria* and 'Sui fondamenti della Geometria', respectively), and the segment formation operation between two points (ab is the class of points that lie between a and b and is taken as a segment) are the fundamental concepts of Peano's construction of elementary geometry.[8]

[7] Unless a reference to an English translation is included after a slash, all quotations from the sources are translated by the author. Page numbers refer to the most recent edition of the source or translation listed in the Bibliography.

[8] Although, strictly speaking, the binary segment formation operation is a primitive notion, Peano often refers to it as a ternary relation of incidence between a point and a segment, and represents it as '$c \in ab$' (see [Peano, 1889b, p. 61]). In fact, in [Peano, 1894, p. 119], Peano makes it explicit that instead of reading '$c \varepsilon ab$' as 'c is a point of the segment ab', he prefers to read it as 'c lies between a and b'. Note however that 'ϵ' (or 'ε' in [1894]) is Peano's membership relation symbol and 'ab' is an individual term that refers

The primitive notions of geometry are not defined, but Peano is very clear about the need to provide a secure grounding for them. Peano's claim that the primitive notions are known to any geometer [1894, p. 116] can be linked to his idea that they are intuitive [1891a, p. 67]. In fact, Peano states that they must be acquired from experience [1894, p. 119] and that their properties are "experimentally true" [1889b, p. 56].

Besides the requirement that the primitive notions be acquired from experience, Peano also considers some methodological principles that are involved in the selection of these concepts. Attributing simplicity to the primitive notions is coherent with the idea that any other geometrical concept has to be defined in terms of them. In addition, precision and the reduction of the number primitive notions to the smallest possible are some of the most most explicit methodological principles in Peano's presentations of logic, geometry or arithmetic (see, for instance, [1889a, p. 21]/[1973, p. 102], [1889b, p. 78] and [1895, pp. 191–192]/[Dudman, 1971, pp. 28–30]).

Relying on an undisputed intuitive basis, simplicity, minimality, and precision guide Peano's selection of primitive notions. In 'Sui fondamenti della Geometria', he rules out the possibility of assuming the notion of space as primitive [1894, p. 117]. In Peano's view, the notion of space is not, strictly speaking, necessary, and such an assumption moreover requires us to add further primitive notions that constitute space's common attributes, namely homogeneity, infinitude, divisibility, immobility, etc., which goes against the criterion of simplicity. Besides, the notion of line, surface and solid are not precise enough for a systematization of the intuitive basis of geometry, and thus are too indeterminate to be considered primitive [1894, pp. 117–118]. Instead, Peano proposes using the notions of straight line, plane and specific solid figures, since they can be defined in terms of classes of points and segments.

In 'Sui fondamenti della Geometria', Peano takes pride in having constructed projective geometry with two primitive concepts, that is, one less than those of Pasch's presentation:

> Pasch, in his important book *Vorlesungen über neuere Geometrie* (Leipzig, 1882), developed Projective Geometry [*Geometria di Posizione*] assuming only three primitive concepts, namely the *point*, the *rectilinear segment* and the finite portion of a *plane*. But the third of these concepts can be reduced to the previous ones by assuming as the definition of the plane, or a part of it, one of its well-known generations [*generazioni*]. Therefore, having admitted the two concepts, point and rectilinear segment, we can define all the other entities, and develop the whole Projective Geometry [*Geometria di Posizione*]. [Peano, 1894, p. 119]

Peano pays much attention to definitions in his construction of geometry, and to the fact that any derived notion can be nominally defined by means of primitive notions using logical symbolism. The formal resources provided by the language of his mathematical logic are instrumental in the formulation of precise and rigorous definitions. However, Peano does not develop a

---

to the result of applying the segment formation function to a and b. See [Marchisotto, 2011].

systematic account of the indefinability of the primitive notions. Such an account would prove to be an important issue in Peano's close mathematical environment: in 'Essai d'une théorie algébraique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque' [1901], Padoa informally characterizes the indefinability—in his terms, irreducibility—of a system of primitive notions with respect to a set of postulates.[9]

Despite the methodological principles that guide the establishment of a set of basic concepts, Peano acknowledges that there is some degree of arbitrariness in his selection. In the context of a specific theory, as long as the primitive notions make it possible to define all derived notions, there is no need to rely on a specific choice. According to Peano, if by means of a and b we can define c, and by means of a and c we can define b, then it is just a matter of preference to decide whether a and b, or a and c are the primitive notions [1889b, p. 78]. Nevertheless, this arbitrariness has its limits. First, as Peano puts it in *Principii di Geometria*, "the signs **1** and $a'b$[10] (point and ray) could have been assumed instead of the signs **1** and ab (point and segment); this would not have been possible assuming the point and the straight line as undefined concepts" [1889b, p. 78].[11]

Second, Peano's remarks on arbitrariness are framed in a single theory—specifically, elementary geometry. Assuming the intuitive basis from which geometry is constructed, Peano does not seem to consider the possibility of building different geometries which might have conflicting sets of primitives. Late nineteenth-century empiricism in geometry is nuanced with respect to the role of intuition in the basic components of geometrical theories. In this regard, Peano's account diverges from Klein's. In a lecture delivered in September 2, 1893, Klein distinguishes between naïve intuition, which is inexact, and refined intuition, which comes as the result of an axiomatization [1911, pp. 41–42]/[Ewald, 1996, II, p. 959]. In Klein's view, the inexactitude of spatial naïve intuition can be organized and systematized in different ways, and can actually form the foundation of different and equally justified

[9] I am indebted to an anonymous referee for bringing Padoa's account of the irreducibility of primitive notions into my attention.

[10] The ray function $'$ determines the class of points that lie beyond a point b relative to a point a. In 'Sui fondamenti della Geometria' [1894, p. 120], Peano defines $a'b$ as follows:

$$a, b, c \,\varepsilon\, \mathrm{p} \,.\, \supset : c \,\varepsilon\, a'b \,.=.\, b \,\varepsilon\, ac.$$

Note that, using $a'b$, ab could be defined:

$$a, b, c \,\varepsilon\, \mathrm{p} \,.\, \supset : c \,\varepsilon\, ab \,.=.\, b \,\varepsilon\, a'c.$$

See also [Peano, 1889b, §2, p. 61, Prop. 1].

[11] In 'Sui fondamenti della Geometria' [1894, p. 126], the concept of straight line (in Italian, *retta*) is defined as follows:

$$a, b \,\varepsilon\, \mathrm{p} \,.\, a \mathbin{-=} b \,.\supset .\, \mathrm{retta}(a, b) = b'a \cup \iota a \cup ab \cup \iota b \cup a'b,$$

where ιa is the class of objects that are equal to a (i.e., the singleton of a).

Following [Moore, 1902, p. 144], Marchisotto [2011, p. 163] suggests that not only simplicity is behind Peano's choice of the segment formation operation as a primitive notion; the notion of segment is more fundamental than the concept of line with respect to a set of postulates based on spatial intuition. In their view, the fundamentality of the notion of segment also played a role in Peano's choice.

geometries [1890, p. 572].[12] Peano does not draw such a distinction on intuition, and he does not suggest that the intuitive content from which the primitive notions of geometry are extracted is inexact. After all, as he states in 'Sui fondamenti della Geometria', the primitive notions are known by anyone who is familiar with geometry, and must already have terms that refer to them [1894, p. 116]. The concepts of point and straight segment constitute, with the axioms, the basis of Peano's construction of elementary geometry. The same intuitive foundation remains for any specific theory derived from elementary geometry, including projective geometry.[13]

Assuming that the primitive notions cannot be defined, Peano refuses to even offer descriptions or elucidations about their nature. In *Principii di Geometria*, he affirms that concerning the primitive notions,"only [their] properties will be stated" [1889b, p. 78]. These properties are expressed in the axioms. As Peano puts it in 'Sui fondamenti della Geometria':

> [I]t will be necessary to determine the properties of the undefined entity p [point], and of the relation $c \varepsilon ab$ [c lies between a and b], by means of axioms or postulates. The most elementary observation shows us a long series of properties of these entities; we just have to collect these common notions [*cognizioni*], order them, and enunciate as postulates only those that cannot be deduced from simpler ones. [Peano, 1894, p. 119]

Peano's remarks that the primitive notions of geometry are acquired from experience, and that the axioms are the result of a systematization of the properties of the fundamental concepts, stand at the core of his construction of geometry. The combination of his specific choice of primitive notions and axioms constitute an analysis of the intuitions of space. This intuitive basis is selected, rearranged and regimented following, as we have seen, methodological criteria. The adoption of the axiomatic method plays a crucial role in this analysis, as it makes possible to systematically collect the most elementary properties of the notions of point and straight segment and build geometry in such a way that the deductive dependencies between axioms and theorems are made explicit.

Although Peano states that the axioms of geometry express the simplest properties of the primitive notions, they cannot be considered explicit definitions of these concepts. As stated above, the primitive notions are left undefined and geometry has to be constructed from axioms. Accordingly, although there is a close connection between the content of the axioms and the nature of the notions of point and straight segment, the former do not

[12]I am indebted to an anonymous referee for suggesting me to consider Klein's account of intuition.

[13]In [1889b] and [1894], Peano's goal is to put forward a synthetic construction of geometry, one that does not rely on any non-geometrical notion. This could be seen as a specific way of systematising the kind of intuition that is relevant in geometry. However, Peano adopts an alternative way of systematising intuitive content in his work on the geometrical calculus (see, for instance, [1888] and [1898]). The geometrical calculus establishes a linear algebra and ultimately rests on the notion of number. On Peano's two ways of organising spatial intuitions, see [Rizza, 2009, p. 357]. On the relationship between Peano's geometric calculus and his synthetic axiomatization of elementary geometry, see [Gandon, 2006].

completely determine the latter. As Peano states, the axioms articulate a selection of the properties of the primitive notions, and as we will see in Section 4, there are multiple systems which can share the structural features stated in the axioms.[14] This specific relationship between the primitive notions and the axioms paves the way for an abstract understanding of the latter. I will consider such an understanding in Section 4.[15]

That said, Peano is not interested in constructing geometry as an abstract theory. The axioms must be founded on direct observation. Such a connection between the axioms and intuitive content is what makes them truly geometrical. In Peano's words:

> [A]nyone is allowed to allow those hypotheses that they want, and develop the logical consequences contained in those hypotheses. But for this work to deserve the name of Geometry, those hypotheses or postulates must express the result of the simplest and most elementary observations of physical figures. [Peano, 1894, p. 141]

As the result of an analysis of spatial intuition, the axioms of geometry articulate the basic properties of the three-dimensional space. Of the sixteen axioms of elementary geometry that are formulated in *Principii di Geometria*, axioms XV and XVI bear witness to Peano's empiricist stance:[16]

$$\text{(XV)} \qquad p \,\epsilon\, \mathbf{3} \,.\, \supset \therefore a \,\epsilon\, \mathbf{1} \,.\, a \,-\!\epsilon\, p : -\!=_a \Lambda.$$

$$\text{(XVI)} \quad p \,\epsilon\, \mathbf{3} \,.\, a \,\epsilon\, \mathbf{1} \,.\, a \,-\!\epsilon\, p \,.\, b \,\epsilon\, a'p \,.\, x \,\epsilon\, \mathbf{1} : \supset : \\ x \,\epsilon\, p \,.\cup.\, ax \cap p \,-\!= \Lambda \,.\cup.\, bx \cap p \,-\!= \Lambda.$$

According to Peano, Axiom XV can be read as "Given a plane, there are points that are not contained in it", and Axiom XVI, "Given a plane, and two points from opposite sides of the plane, either each point of space lies on the given plane, or one of the segments that connect it to the given points meets the plane" [1889b, p. 89]. Peano concludes that Axiom XVI states that the space is three-dimensional. Although, as we will see in the next section, Peano considers the possibility of a higher-dimensional space, he does not include any axiom in his construction of elementary geometry that postulates the existence of high-dimensional spaces. In fact, as we will see in Section 3.3, Axiom XVI would have to be dropped in an axiom system of a

[14] On a structuralist understanding of Peano's axiomatization of geometry, see [Bertran-San Millán, 2022].

[15] Rizza [2009] suggests a similar idea. In his words:

> [T]he need to systematically organize spatial intuition around certain fundamental concepts can give rise to the concept of a formal structure as a type of organization of a given intuitive content. The choice of fundamental concepts and the articulation of geometry on their basis is carried out through the axiomatic method. [Rizza, 2009, p. 366]

[16] Note that **3** is the class of classes of points that form a plane; Λ, depending on the context, is the empty set (Axiom XVI) or a propositional constant that means the absurd (Axiom XV); and a formula that contains an equality symbol with a letter attached to it as a subscript is the universal quantification of a biconditional.

four-dimensional space. Had Peano understood his axiom system as a purely abstract structure, this limitation would not be justified.[17]

Furthermore, in 'Sui fondamenti della Geometria', Peano considers the proposition "Two straight lines lying in the same plane always have a point in common" as a possible axiom of projective geometry. He rejects such a possibility because this proposition is "not verified by observation, and it is indeed in contradiction with Euclid's theorems" [1894, p. 141]. As Peano states, "projective Geometry originates from the postulates of elementary Geometry and, by means of appropriate *definitions*, it introduces new entities, called ideal points (both in Euclidean and non-Euclidean geometry)" [1894, p. 149]. He explicitly claims that by means of these new entities all the axioms of elementary geometry are satisfied. All in all, for Peano projective geometry is derived from elementary geometry through definitions, and thus all the axioms of the former must be confirmed by direct observation.[18]

In *Principii di Geometria* [1889b, pp. 84–85], Peano analyses the content of three of Pasch's axioms from *Vorlesungen über neuere Geometrie* [1882] and establishes correspondences between his axioms of linear geometry and Pasch's. In 'Sui fondamenti della Geometria' [1894, p. 120] Peano again acknowledges that his axioms of linear geometry essentially correspond to Pasch's.[19] Besides the postulates of linear geometry, Peano also shares with Pasch the requirement of an empiricist foundation of geometry.[20] Pasch's empiricism is idiosyncratic, but commonalities with Peano's account can nonetheless be found. In *Vorlesungen über neuere Geometrie* [1882, p. 3], Pasch claims that geometry is a natural science. He also offers a characterization of the basic concepts that echoes Peano's:

> *The basic concepts* [*Grundbegriffe*] *are not defined*; no explanation is able to replace that means which alone eases the understanding of those simple concepts that cannot be traced back to others, namely the reference to suitable physical objects [*geeignete Naturobjecte*]. [Pasch, 1882, p. 16]

As we will see in the next section, Peano also shares the reservations expressed by Genocchi—with whom Peano collaborated as assistant during the first years of the 1880s—concerning a purely abstract foundation of geometry.

---

[17]On the idea that Peano axiomatizes the properties of a three-dimensional space, see [Rizza, 2009, pp. 362–364].

[18]Similarly, Pasch reflects on the addition of an axiom of continuity, but then rejects such a possibility on the grounds that it is inconsistent with his empiricist stance [1882, pp. 125–127]. I am indebted to an anonymous referee for suggesting me to consider Pasch's reflection on the axiom of continuity.

[19]See [Gandon, 2006, pp. 284–287] for a comparison between Pasch's [1882] axioms of projective geometry and Peano's [1889b] axioms of elementary geometry. See also [Borga *et al*, 1985, pp. 206–211].

[20]On Pasch's empiricism and, in general, on his philosophy of mathematics, see [Schlimm, 2010]. On Pasch's influence in Peano's axiomatization of geometry, see [Borga *et al*, 1985, pp. 52–54]. Gandon [2006] offers an alternative account of Peano's empiricism and its relationship with Pasch's *Vorlesungen über neuere Geometrie*.

3. Peano's critique of Segre's geometry of hyperspaces

Although there is textual evidence concerning Peano's position on the foundations of geometry, his views can be better understood if they are juxtaposed with alternative conceptions of the basis of this mathematical theory. Peano's empiricism can thus be put into an explanatory context, especially on those occasions when he criticizes a purely abstract foundation of geometry. In fact, Peano's criticism is instrumental in understanding the role of an empirical foundation rather as a guiding principle in the axiomatization of geometry than an ad-hoc imposition. Moreover, he makes an effort to explain his views on the abstract character of geometrical proofs when he detects that certain mathematical reasoning lacks rigour. On those occasions, Peano substantiates the claim that, in addition to this empirical foundation, there is a stage in the construction of geometry where it can be understood as an abstract discipline. The study of Peano's polemical exchange with Segre will serve as a transition between my accounts of the former's empiricism and the abstract nature of mathematical proofs.

3.1. **Segre's hyperspace geometry.** As one of the driving forces behind the Italian school of algebraic geometry, Segre was highly influential in the popularization of Klein's *Erlangen program* in Italy.[21] He also made important contributions to hyperspace projective geometry and algebraic geometry. For the purposes of this paper, I will focus on Segre's work on the foundations of hyperspace geometry, which was heavily influenced by the works of Clebsch, Veronese and D'Ovidio.[22] Segre did not follow the axiomatic method and his foundational work on geometries of n-dimensions was constructed upon an abstract notion of point.

In 'Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni' Segre introduces the notion of point as follows:

> Let us consider any linear space of $n-1$ dimensions. We will call *point* each of its elements, whatever their nature (which is of no importance to us). [Segre, 1883, p. 39]

A point is presented just as an n-sequence of real numbers and Segre rejects any reflection upon its nature. In fact, Segre dismisses intuitions of space and, as a consequence, all linear spaces of a given number of dimensions are identified:

> All linear spaces with the same number of dimensions, whatever their elements are, can be regarded as identical to each other, since, as we have already noted, in studying them the nature of those elements is not considered, but only the property of linearity and the number of dimensions of the space formed by the elements themselves. [Segre, 1883, p. 46]

Although Segre's characterization of a linear space [1883, p. 38] does not meet contemporary standards of rigour (nor, in reality, even Peano's)[23], its

---

[21]On Segre's leadership of the Italian school, see [Conte; Giacardi, 2016] and [Luciano; Roero, 2016].

[22]On Segre's contributions to the foundations of geometry, see [Brigaglia, 2016].

[23]On a comparison between Segre's and Peano's definitions of a linear space, see [Avellone *et al*, 2002, pp. 375–377].

abstract character is fundamental to the incorporation of algebraic tools into geometry and, the characterization of the relationships between linear spaces of different dimensions. It is at the essence of Segre's notion of linear space that, as he puts it in in 'Su alcuni indirizzi nelle investigazioni geometriche', "every space is contained in a higher one; and in the latter we may seek for forms which will simplify the study of given forms in the former" [1891a, p. 63; 465].[24]

Segre published a long paper addressed to students, 'Su alcuni indirizzi nelle investigazioni geometriche' [1891a], in the first volume of *Rivista di matematica*. Despite the introductory and general character of the paper, it triggered an unusual response from Peano, who was the editor and one of the founders of the journal. Peano placed his 'Osservazioni del Direttore sull'articolo precedente' [1891a] immediately after Segre's paper in the same volume of *Rivista di matematica*. Segre's reply [1891b] was also published, and this in turn prompted Peano's final reaction [1891b].[25]

The dispute sparked by Peano is mainly concerned with mathematical rigour and the use, in geometrical works, of principles lacking solid demonstration. However, Peano also criticizes Segre's construction of hyperspace geometry, and this will be the focus of my discussion in this section. In particular, I will consider the two main aspects of Peano's critique: on the one hand, the lack of empirical character of basic propositions and primitive notions of a foundational work on geometry; and, on the other, the unjustified analogical use of $n+1$-dimensional geometry to obtain results of n-dimensional geometry.

3.2. **The abstract foundation of hyperspace geometry.** In 'Osservazioni del Direttore sull'articolo precedente', Peano insists on some ideas that he had suggested in *Principii di Geometria* and would develop in 'Sui fondamenti della Geometria'. Specifically, in his first reaction to [Segre, 1891a], Peano puts forward his claim concerning the empirical character of the axioms of geometry. He suggests that geometry cannot be built upon "hypotheses contrary to experience, or [...] hypotheses which cannot be verified by experience" [1891a, p. 67].

Peano then elaborates on this view and suggests that there is a pre-mathematical phase in which the axioms are selected and formulated:

> Each author can assume those experimental laws that they please, and can make those hypotheses that they like best. The good choice of these hypotheses is very important in the theory to be developed; but this choice is made by way of induction, and does not belong to mathematics. Having made the choice of the starting point, it is up to mathematics (which, in our opinion, is a perfected logic) to deduce the consequences; and these must be absolutely

[24]It is worth mentioning that other prominent members of the Italian school of algebraic geometry did not share Segre's point of view and argued for empiricism. Veronese, whose work on hyperspace geometry influenced Segre, advocated for using empirically-grounded basic concepts [1891, pp. 611–612].On Veronese's work on the foundations of geometry, see [Cantù, 1999]. See also [Avellone *et al*, 2002, pp. 380–385].

[25]On the polemic between Peano and Segre, see [Manara; Spoglianti, 1977], [Borga *et al*, 1985, pp. 242–244], [Bottazzini, 2001, pp. 553–555], [Avellone *et al*, 2002, pp. 372–385].

> rigorous. Whoever states consequences that are not contained in the premises might make poetry, but not mathematics. [Peano, 1891a, p. 67]

These remarks complement the picture laid out in the previous section concerning the establishment of the axioms of geometry. For Peano, the foundation of geometry begins with a stage where the primitive notions are selected. The properties of these primitive notions are obtained by direct observation, and they are rearranged and systematized in a list of axioms. The axioms can be understood as experimental because they state the basic properties of the primitive notions, which are obtained from experience. Therefore, in Peano's view, the result of direct observation is not imposed upon a set of abstract axioms at a later stage; it is inherent in these axioms that they select, rearrange and regiment intuitive content. Once this pre-mathematical analysis has produced a specific list of axioms, it is followed by mathematics proper, which consists in the definition of derived notions and the demonstration of theorems. In the next section I will evaluate Peano's claim that mathematics is "a perfected logic".

With these assertions alone, Peano is ready to discredit Segre's foundations of hyperspace geometry, viewing them as not genuinely geometrical. If a point is characterized just as an n-sequence of numbers, the intuitive character attached to this concept is completely lost. Moreover, the primitive notions of geometry are no longer independent of the notion of number and thus the boundaries between geometry and analysis—which relies on the concept of number—become blurred.[26] In Peano's words:

> If any group of n variables is called a point [. . .], then it is well known that any discussion on the postulates of Geometry ceases; the theories that are deduced develop the consequences of the principles of arithmetic, and not of those of geometry; every result thus obtained is independent of any geometric postulate. [Peano, 1891b, p. 157]

Peano advocates for an autonomous foundation of geometry, one which does not rely on non-geometrical notions. This is coherent with his synthetic approach in the construction of geometry, and implicitly encapsulates an idea of *purity of method*.[27] For Peano, Segre's foundation of hyperspace geometry is not pure and, moreover, lacks an account that connects the basic concepts with our intuitions of space.[28]

---

[26]As Rizza [2009, p. 357] suggests, Peano does not rule out n-dimensional linear spaces because they are used in ordinary mathematics; he does not accept them *in geometry*, since their existence is not supported by our intuitions of space.

[27]On the notion of purity of method, see [Arana, 2008], [Detlefsen, 2008], [Detlefsen; Arana, 2011].

[28]Peano's empiricism and his critique of Segre's abstract foundation of n-dimensional geometries can be connected with the views of Genocchi, Peano's predecessor as the chair of infinitesimal calculus in Turin. In [1891, pp. 614-615, fn. 2], Veronese reports Genocchi's dismissive and harsh judgement of hyperspace geometry, which can be found in [Genocchi, 1877, pp. 388–389]. On Genocchi's views of hyperspace geometry and the polemic between Peano and Segre, see [Manara; Spoglianti, 1977].

3.3. **An axiomatic construction of hyperspace geometry.** Let us now turn to the second aspect of Peano's critique of Segre's construction of hyperspace geometry. In 'Su alcuni indirizzi nelle investigazioni geometriche', Segre suggests three possible foundations of hyperspace geometry, which in turn correspond to three possible ways of defining points in an n-dimensional linear space [1891a, pp. 59–61]/[1904, pp. 460–463]. The first is the one already considered, and takes points to be "any system of values of n variables (the *coördinates* of the point)" [1891a, p. 59]/[1904, p. 460]. The second follows Plücker and characterizes points as "geometric forms of ordinary space, such as groups of points, curves, surfaces" [1891a, p. 60]/[1904, p. 461]. Finally, according to the third option, points in hyperspace are characterized as ordinary points, but "we omit the postulate concerning the three dimensions, and consequently modify some of those referring to the straight line and plane" [1891a, p. 60]/[1904, p. 462].

Concerning the first option, Segre already anticipates Peano's critique that it results in an algebra of linear transformations and it is thus no longer genuine geometry [1891a, p. 59]/[1904, p. 461]. However, he makes it clear that this is not an issue for him, since, after all, "it is *mathematics* that is being made" [1891a, p. 59]/[1904, p. 461, fn. 2]. In his response to Segre, Peano only considers Segre's third possible foundation, and it is on this matter that he levies his critiques.

Peano describes his proposal of an axiomatic construction of a four-dimensional geometry as follows:

> To move from the 3-dimensional space to the 4[-dimensional space], it is necessary to eliminate the 16th postulate, and then, without modifying those referring to the straight line and the plane, to admit the postulate, analogous to [postulates] 2, 7, 12, 15:
>
> *A*] *There are points outside ordinary space.*
>
> It follows as a consequence [. . .] that, in this way, *every proposition proved true using the 4-dimensional space ceases to hold in the 3-dimensional space*, since it is shown to be a consequence of postulates 1–15 and postulate A, and it is not shown to be a consequence of the postulates of elementary geometry alone. [Peano, 1891a, p. 68]

In *Principii di Geometria*, sixteen axioms establish the basis of elementary geometry.[29] Peano suggests axiomatizing the four-dimensional space by means of axioms I–XV and axiom A. In the aforequoted passage, he refers

[29] In an Appendix, Peano also formulates a seventeenth axiom which postulates the continuity of the straight line [1889b, p. 90].

to Axioms II, VII, XII and XV:[30]

$$\text{(II)} \qquad a \,\epsilon\, \mathbf{1} \,.\, \supset \therefore x \,\epsilon\, \mathbf{1} \,.\, x \mathrel{-\!\!=} a : \mathrel{-\!\!=}_x \Lambda.$$

$$\text{(VII)} \qquad a, b \,\epsilon\, \mathbf{1} \,.\, a \mathrel{-\!\!=} b : \supset .\, a'b \mathrel{-\!\!=} \Lambda.$$

$$\text{(XII)} \qquad r \,\epsilon\, \mathbf{2} \,.\, \supset \therefore x \,\epsilon\, \mathbf{1} \,.\, x \mathrel{-\epsilon} r : \mathrel{-\!\!=}_x \Lambda.$$

$$\text{(XV)} \qquad p \,\epsilon\, \mathbf{3} \,.\, \supset \therefore a \,\epsilon\, \mathbf{1} \,.\, a \mathrel{-\epsilon} p : \mathrel{-\!\!=}_a \Lambda.$$

Axiom II states that given any point a, there are points different from a. Axiom VII states that given two points a and b, if they are different, then the ray $a'b$ is non-empty (and thus there are points which lie in $a'b$). Axiom XII states that given a line r, there are points which do not lie on r. As indicated in the previous section, Axiom XV states that given a plane p, there are points which do not lie on p. These axioms are all existential and thus, as Peano states, analogous to the suggested axiom A; they postulate the existence of points that do not meet certain conditions.

Then, as a means of axiomatizing a four-dimensional space, Peano also proposes eliminating Axiom XVI:[31]

$$\text{(XVI)} \quad p \,\epsilon\, \mathbf{3} \,.\, a \,\epsilon\, \mathbf{1} \,.\, a \mathrel{-\epsilon} p \,.\, b \,\epsilon\, a'p \,.\, x \,\epsilon\, \mathbf{1} : \supset : \\ x \,\epsilon\, p \,.\, \cup \,.\, ax \cap p \mathrel{-\!\!=} \Lambda \,.\, \cup \,.\, bx \cap p \mathrel{-\!\!=} \Lambda.$$

According to the construction put forward by Peano, any theorem that is demonstrated by means of the axiom system of a four-dimensional space cannot be considered a theorem of a three-dimensional space, since it has not been proved from axioms I–XVI. After all, if a theorem is deduced from axioms I–XV and A, then it cannot be considered a theorem of elementary geometry proper, since axiom A can play a role in its proof. Peano's argument attempts to block Segre's strategy, according to which results obtained in $n+1$-dimensional linear spaces can be applied to n-dimensional spaces; the inclusion of axiom A in Peano's construction involves a a substantial use of $n+1$-dimensional tools.[32]

Peano's conclusion is that Segre's analogical use of four-dimensional linear spaces to prove theorems of three-dimensional linear spaces is unjustified. In his words:

> Some writers, from the fact that many properties of plane figures are derived from properties of solid figures, deduce by analogy that properties of figures of ordinary space can be derived from

[30] Note that **2** is the class of classes of points that constitute straight lines, and recall that **3** is the class of planes, and Λ, depending on the context, is the empty set (axiom VII) or a propositional constant that means the absurd (axioms II, XII and XV). See Footnote 16.

[31] Note that, in this context, Λ is the empty set. See Section 2 for an informal rendering of Axiom XVI.

[32] In my view, Peano's argument focusses on the fact that a theorem demonstrated in a four-dimensional space is *unjustified* in a three-dimensional space, and thus relies on its epistemological status rather than on its being true or false in a three-dimensional space. Bottazzini [2001, pp. 303–304] reports an alternative interpretation of the aforequoted passage found in [Bozzi, 2000, pp. 104] and suggests that Peano might identify theory and interpretation.

> considerations in 4-dimensional space. But the analogy is illusory. [Peano, 1891a, p. 68]

A corollary of Peano's statement would be that, if the use of four-dimensional space in the proof of a three-dimensional theorem cannot be taken for granted, then the fact that "many properties of plane figures are derived from properties of solid figures" is also unjustified for similar reasons. Peano's conception of what constitutes a specific geometry, which can be seen as an defence of purity of method of proof, clarifies this issue.

3.4. **Purity and Desargues's theorem.** In the first reply to Segre's [1891a] paper, Peano argues that linear, planar and solid geometry are constituted by specific axioms:

> [I]f by geometry of the straight line (1-dimensional) we mean that which develops the consequences of axioms 1–11; by plane geometry (2-dimensional) that which develops the consequences of [axioms] 1–14, and by solid geometry that which also uses the 15th axiom, then we will have drawn a scientific distinction. [Peano, 1891a, p. 68]

Informally, it could be observed that some geometrical propositions deal with straight lines and segments, others with plane figures and others with solid figures. But, as Peano states, this is just a "didactic distinction" [1891a, p. 68]. In his view, the axioms from which a theorem is deduced, rather than its informal content, determine its nature. Therefore, a proposition whose proof requires the use of axioms of linear, planar or solid geometry will be considered a theorem of linear, planar or solid geometry, respectively, even if what is suggested by its wording indicates otherwise.

Peano's criterion for the stratification of elementary geometry can be connected to the debate on fusionism that took place in the last years of the nineteenth century. The debate focussed on the use of solid geometry in the solution of problems and the proof of theorems of planar geometry.[33] In fact, Peano played a significant role in this debate with his contribution to the clarification of the status of Desargues's theorem on homological triangles—which he called '*teorema fondamentale sui triangoli omologici*'. According to its planar version, Desargues's theorem states that if the corresponding vertices of two triangles that lie on the same plane intersect in a point, then the intersections of the corresponding sides of the two triangles are collinear. Desargues's theorem occupied a prominent place in the debate on fusionism because, even though the content of its planar version suggests that it belongs to planar geometry, its proof uses solid techniques.

In *Principii di Geometria*, Peano states that from Axiom XV and the following theorem:

$$p \,\epsilon\, \mathbf{3} \,.\, a, b, c, d \,\epsilon\, p \,.\, a \,{-}{=}\, b \,.\, c \,{-}{=}\, d \,.\, e \,\epsilon\, \mathbf{1} \,.\, e \,{-}\epsilon\, p : \supset \therefore \\ r \,\epsilon\, \mathbf{2} \,.\, r \supset (eab)'' \,.\, r \supset (ecd)'' : {-}{=}_r \, \Lambda,$$

which can informally be read as "if in a plane p there lie two straight lines ab and cd, and if e is a point outside the plane p, then the planes $(eab)''$ and $(ecd)''$ have a line in common", Desargues's theorem can be proved

[33]On the debate on fusionism, see [Arana; Mancosu, 2012, pp. 302–324].

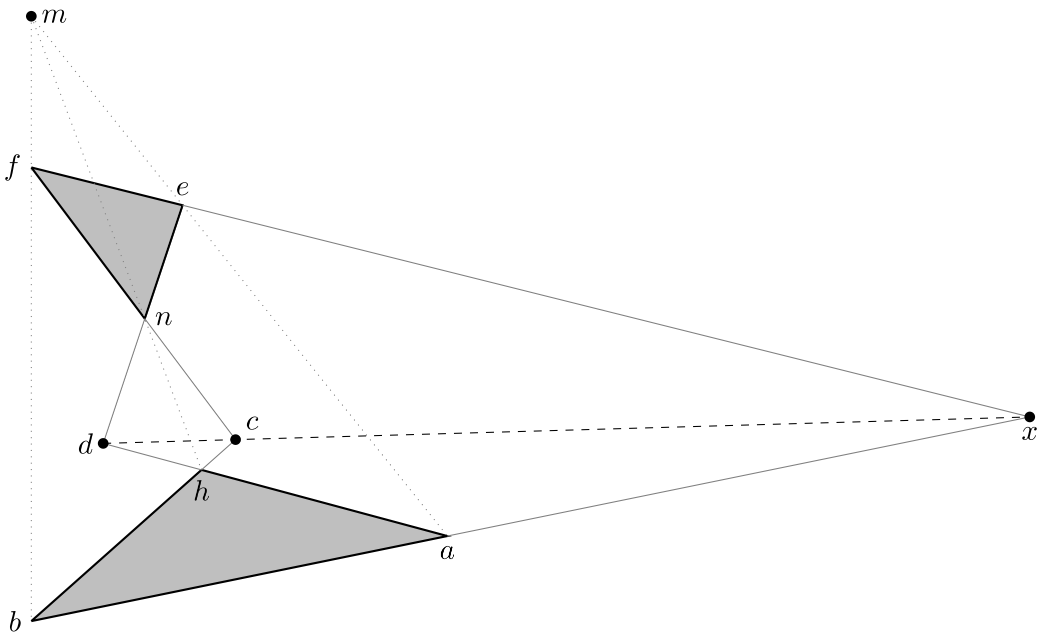

FIGURE 1. Planar Desargues's theorem adapted from Peano's [1894, p. 139] formulation.

[1889b, p. 89]. In 'Osservazioni del Directore sull'articolo precedente', Peano confirms this idea, but he also adds the following:

> [T]he geometry of the straight line is reduced to almost nothing [...]. The geometry of the plane is already broader; the coordinates can already be established there, but neither the equation of the straight line can yet be found [...], nor can one demonstrate the theorem of homological triangles. [Peano, 1891a, p. 68]

Although he does not specify as much, it is reasonable to assume that Peano refers here to the planar version of Desargues's theorem. He argues, without justification, that this theorem is independent of planar geometry. This is historically significant, since, despite the informal content of the planar version of the theorem, an independence result settles the impossibility of proving it using exclusively planar means. In 'Sui fondamenti della Geometria', Peano formulates the solid version of Desargues's theorem as follows (see the corresponding planar version in Figure 1):

> If among the ten points e, a, b, c, d, h, m, n, x, the first four of which are not coplanar, nine occur in the following relations:
>
> $$h \,\varepsilon\, ad \,.\, h \,\varepsilon\, bc \,.\, e \,\varepsilon\, am \,.\, n \,\varepsilon\, ed \,.\, n \,\varepsilon\, mh \,.\, f \,\varepsilon\, mb \,.\, n \,\varepsilon\, cf \,.\, a \,\varepsilon\, xb \,.\, c \,\varepsilon\, xd \,.\, e \,\varepsilon\, xf,$$
>
> then the remainder will also occur in them. [Peano, 1894, p. 139]

On this occasion, Peano explicitly distinguishes between the planar and the solid versions of Desargues's theorem, and argues that both can be demonstrated from the aforementioned theorem and axiom XV [1894, p. 139]. And then he goes on to state the following:

> The theorem of homological triangles in the plane is, however, a consequence of postulate XV, and therefore it is a theorem of solid

> geometry. That it is not a consequence of the previous postulates is shown [by the following:] if by p we mean the points of a surface, and by $c \varepsilon ab$ we mean that the point c lies on the geodesic arc that joins the points a and b, then all the postulates from I to XV are verified, and the proposition on homological triangles does not always hold [*non sussiste sempre*]. However, this proposition continues to be valid for surfaces with constant curvature. [Peano, 1894, p. 139]

This is a confirmation of the claims made in 'Osservazioni del Directore sull'articolo precedente' that Desargues's theorem belongs to solid geometry and is independent of planar geometry. Peano sketches an independence proof by exemplification—that is, he gives an example of an interpretation of the primitive terms that satisfies all the axioms of planar geometry (namely, axioms I–XIV) but does not satisfy Desargues's theorem. Peano thus implicitly, and for the first time, considers the possibility of a non-Desarguesian plane.[34] In the previous section, it has been observed that Peano advocated purity concerning his preference for the autonomy of geometry from the concept of number and his rejection of the use of analysis in the foundations of geometry. There is a second notion of purity involved in Peano's (sketched) proof of independence of Desargues's theorem from the axioms of plane geometry. This second notion deals with the method of proof; the proof of a theorem is considered pure if it does not require any other means than what is stated in the theorem. Purity of method and purity of method of proof are related in the sense that they involve the avoidance of concepts which are foreign to the relevant question. Both notions of purity are also historically connected; as Arana and Mancosu state, in the fusionist debate regarding the use of solid techniques in the solution of planar problems " 'purity' in the direction of eliminating metrical considerations from proofs of projective theorems might come at the cost of bringing considerations related to space in poofs of plane theorems" [2012, p. 303]. Peano's claim that Segre's conception of point is arithmetical can be associated with a rejection of metrical considerations in the foundation of geometry. Moreover, Peano's view of the content of geometrical statements, and specifically of the theorems of solid geometry (even if their informal content indicates that they are planar) constitutes his way out of the issue suggested by Arana and Mancosu.[35]

Peano's stratification of geometry on the basis of groups of axioms anticipates his account of the nature of Desargues's theorem. Despite the

[34]For a reconstruction of Peano's sketch of a proof of independence and a proposal of a suitable model, see [Arana; Mancosu, 2012, pp. 317–321].

That Peano, and not Hilbert, was the first to consider a non-Desarguesian plane is not unanimously acknowledged. While Whitehead [1906, p. 11] defends that Peano, and then Hilbert, proved the consistency of a non-Desarguesian plane, Hallett [2008, p. 225] mentions only Hilbert as the first who considered a model of non-Desarguesian geometry. The model suggested in [Arana; Mancosu, 2012, pp. 317–321] can help to settle this historical issue; I agree with their claim that "Peano deserves priority for having first found a non-Desarguesian plane" [2012, p. 323].

[35]I am indebted to an anonymous referee for suggesting that there are two different notions of purity involved in Peano's critique of Segre's hyperspace geometry.

fact that the informal content of the planar version involves only planar considerations, for Peano—since its proof requires the use of axiom XV and thus solid geometry—it has to be considered a theorem of solid geometry. In his words:

> [A] true proposition in plane geometry ceases to hold [*cessa di sussistere*] in the geometry of the straight line, and a proposition of solid geometry no longer holds [*non sussiste più*] in plane geometry. The theorem of homological triangles is then a proposition of solid geometry and not of plane geometry. [Peano, 1891a, p. 69]

According to Peano's notion of purity of method of proof, there is no fundamental connection between what a theorem informally states and the principles involved in its proof. In this sense, what is suggested by the wording of the theorem is irrelevant, since its nature is completely determined by its deductive relations with the axioms. As has been stated above, the proof of independence of Desargues's theorem from planar geometry is significant in this regard, because it shows that no proof that uses only planar geometry is possible and, therefore, the possibility of counting this theorem among the propositions of planar geometry is—according to Peano's notion of purity—ruled out.[36]

Peano's notion of content, which is completely determined by deductive relations, signals how he conceives the development of geometry: that is, as the process of demonstration of theorems from the axioms. Peano's considerations regarding Desargues's theorem indicate that this process is, for Peano, purely formal, and that there is no room for any significant appeal to specific geometrical content.

4. Abstract development of geometry

As has been observed in Section 3.2, in the construction process of elementary geometry, Peano distinguishes between a pre-mathematical phase, where the axioms are selected and formulated, and a properly mathematical phase, where the consequences of those axioms are derived. The latter phase corresponds to a "perfected logic" [1891a, p. 67], where absolute rigour is fundamental. Peano argues that every step in a proof has to be determined by rigorous laws and and he thus dismisses any role played by intuition or by any other principle that is logical, or is not included in the axioms or the definitions [1891a, p. 67].

This is confirmed by Peano in *Principii di Geometria*, where he puts forward his conception of mathematical—and specifically geometrical—proof. For Peano, in a demonstration, mathematical laws are put in a form analogous to algebraic equations, and are then grouped and transformed according to laws of reasoning expressed in the form of logical identities [1889b, p. 81]. Peano first evaluated the logical principles that could be used to regiment a mathematical proof in the introductory part of *Arithmetices principia nova*

[36]In contrast to Peano, Hilbert seems to attribute more importance to the informal content of Desargues's theorem. In his 1898–1899 lecture notes, Hilbert states that the content of Desargues's theorem belongs to planar geometry, while its proof requires the use of (three-dimensional) space [Hallett; Majer, 2004, pp. 223, 315–316]. On Hilbert's notion of purity of method, see [Hallett, 2008] and [Arana; Mancosu, 2012, pp. 324–344].

methodo exposita [1889a, pp. 24–33]/[1973, pp. 104–113]. Accordingly, in mathematical demonstrations, it is fundamental that mathematical propositions are expressed in such a way that logical laws can be applied to them. Peano affirms that propositions should be reduced to formulas analogous to algebraic equations [1889b, p. 81], but his symbolization of geometry is actually rather more complex and sophisticated.[37] Using just a minimal collection of primitive geometrical terms (which receive a symbolic representation), Peano expresses any proposition of geometry by means of the formalism of his mathematical logic. The formulation of the axioms of elementary geometry, some of which have been presented in Section 3.2, are examples of Peano's symbolization. No trace of natural language can be found in these axioms; in fact, logical or class-theoretical symbols are used to connect geometrical terms. Peano's symbolization of geometry aims at expressing geometrical laws without ambiguity and with unimpeachable rigour.

Despite Peano's efforts to systematize mathematical proofs, he did not fully develop a deductive calculus. The demonstrations included in *Principii di Geometria* [1889b] and 'Sui fondamenti della Geometria' [1894] are, in fact, sketches of proofs wherein most steps—and the logical laws which regiment them—are not made explicit. Yet, it is clear that, besides the geometrical axioms and definitions that are used as premises in demonstrations, only logical laws can be used as a means to proceed in a proof [1889b, p. 81]. Moreover, Peano developed his calculus of classes and sentential calculus to a significant degree, and incrementally refined the formal apparatus that could be used to regiment mathematical proofs. From the first part of the second volume of the *Formulaire de mathématiques* [1897, p. 254] onwards, Peano provides a list of inference rules. In 'Formules de Logique Mathématique', these inference rules are understood as general rules of reasoning [1900, pp. 320–322], and Peano offers at least one instance of a fully-formalized proof in the calculus of classes, in which every step is the result of the application of an explicitly stated inference rule [1900, pp. 325–327].[38]

In contrast with Pasch or Hilbert, Peano's first works on the foundations of mathematics develop, to a certain extent, a notion of proof and lay down fundamental elements of a fully formalized deductive calculus. The development of logical calculi and efforts toward the symbolization of mathematical statements are key elements in this context. Peano's claim that different geometries are determined by the fact that their theorems are consequences of specific lists of axioms relies on his work on the systematization of the notion of proof.

In fact, Peano's sentential calculus and calculus of classes are fundamental for verifying that only the axioms, the definitions of derived notions, and logical laws play a role in geometrical proofs. Once the axioms and primitive notions are established as an analysis of intuitive content, and all derived notions are defined in terms of the selected primitive concepts, geometry does not rely on the specific nature of these primitive notions. In other words, after a pre-mathematical phase where the empirical foundation of geometry

[37]On the notion of symbolization, and on Peano's reformulation of mathematical theories, see [Bertran-San Millán, 2021b].

[38]On the evolution of Peano's logical calculi, see [Bertran-San Millán, 2021a].

is secured and its basic components are laid down, the specific nature of the fundamental concepts is irrelevant in a second, properly mathematical phase. This abstract character of the axioms comes into play in the derivation of geometrical laws. According to Peano's characterization of the process of demonstration of theorems, the meaning of the primitive terms is left aside. In Peano's words:

> [T]here is a category of entities, called points. These entities are not defined. Moreover, given three points, a relationship between them is considered, expressed by the script $c \epsilon ab$, which likewise is not defined. The reader can understand [*intendere*] by the sign **1** any category of entities, and by $c \epsilon ab$ any relationship between three entities of that category; all the definitions that follow (§2) will always have a value, and all the propositions of §3 will be founded [*sussisteranno*].[39] [Peano, 1889b, p. 77]

Peano's remark that any meaning can be attached to the symbols '**1**' and '$c \epsilon ab$' amounts to his saying that he considers them abstract symbols, that is, symbols with no specific meaning, to be used as place-holders for any instance of a particular category of entities. As Peano states, '**1**' can refer to any domain of entities and '$c \epsilon ab$' to any relation between three of these entities.

Peano's proofs of independence bear witness to the abstract application of such primitive terms. In the previous section, the sketch of the proof of independence of Desargues's theorem from planar geometry has been considered, and there Peano provides an interpretation of the primitive terms that does not correspond to their standard interpretation. In fact, Peano even considers examples of interpretation of the geometrical primitive terms that fall outside geometry, which reinforces the idea that, in certain contexts, these terms may be treated as abstract. Consider the following interpretation of Axiom III found in *Principii di Geometria*:

(III) $$a \epsilon \mathbf{1} \,.\, \supset .\, aa = \Lambda.$$

> If any relation between three entities takes the place of the fundamental relation $c \epsilon ab$, this proposition [Axiom III] is not true in general. If **1** means (finite) number, and we take as the fundamental relation an equation $f(a,b,c) = 0$, which we will suppose algebraic and of first degree in c, the coefficient of c in $f(a,b,c)$ must be divisible by $a - b$, and the known term must not be [divisible by $a - b$], for Prop. 3 [Axiom III] to be true—which here in our case means: the equation $f(a,a,c) = 0$ cannot be satisfied by any value of a and c. [Peano, 1889b, p. 83]

Note that the interpretation provided of the terms '**1**' and '$c \epsilon ab$' determines that Axiom III acquires a completely different meaning. However, this does not invalidate the argument as a consideration of the semantic status of Axiom III. There is no mention of those empirical aspects of the primitive terms that make the content of Axiom III truly geometrical, and yet there is

[39] In §2 of *Principii di Geometria* [1889b, pp. 61–62] Peano defines, among other derived notions, the ray operation $'$ and the classes **2** and **3** of straight lines and planes, respectively. The theorems in §3 [1889b, pp. 62–64] draw consequences from these definitions.

absolutely no doubt that in this passage Peano is establishing conditions of satisfiability of Axiom III.

All in all, the notion of mathematical proof Peano develops, together with his conception of the meaning of the primitive terms of geometry in the context of the demonstration of theorems, can be connected to *deductivism*. Deductivism revolves around the idea of guaranteeing rigour in mathematical reasoning, and became prominent at the turn of the twentieth century after the works of Frege, Pasch and Hilbert. As Pasch argues in *Vorlesungen über neuere Geometrie* with regard to projective geometry, "the process of deducing must everywhere be independent of the *sense* of geometrical concepts; [...] only the *relationships* between the geometrical concepts [...] should come into consideration" [1882, p. 98]. For Pasch, all traces of intuition have to be eliminated in mathematical proofs by disregarding the meaning of the symbols in deductions.[40] Pasch's account squares with Peano's consideration of a propaedeutic phase and a mathematical phase in the construction of mathematical theories, and his claim that, in the later mathematical phase, the meaning of the primitive terms is irrelevant.

The fact that Pasch can be considered an empiricist and a deductivist mathematician and that both aspects are instrumental in his axiomatization of projective geometry help shed light on my reconstruction of Peano's geometry. Peano was in no way alone in his effort to reconcile the selection of a core of empirically-informed concepts and axioms, with a conception of the derivation of theorems that leaves aside the meaning of the primitive terms occurring in them.

5. Concluding remarks

For Peano, geometry had to be constructed from the simplest and the fewest primitive notions possible. This determined to a great extent his choice of the notion of point, and the relation of incidence between a point and a segment, as the basic concepts of geometry. The concepts of line, plane or even the notion of space are either unnecessarily complex—and can be defined in terms of point and segment—or too inexact. Moreover, Peano's resolute preference for synthetic geometry, guided by his conception of purity of method, informs his claim that geometry proper requires its foundations to be free from arithmetical or algebraic considerations, and thus rules out the involvement of analysis or algebra as the basis of the construction of geometry.

In this sense, Peano's disagreement with Segre concerning hyperspace geometry is not only methodological, but also fundamental. To take advantage of the use of results of analysis in geometry, Segre advocates for an abstract notion of point that is determined exclusively as a sequence of real numbers, and thus defends a close relationship between analysis and geometry. In contrast, for Peano the primitive notions must be intuitive and obtained by experience, and then provide the base for the formulation of a collection of axioms.

That Peano insists first, on simplicity, precision, and maximally reducing the number of primitive notions, and second, on the intuitive character

[40] On Pasch's deductivism, see [Schlimm, 2010, pp. 102–107].

of these notions guarantees, to a certain extent, a secure ground for the construction of geometry. But, since these notions cannot be defined and only their relational features are expressed in the axioms, the primitive geometrical notions are underdetermined in the axiomatization. This is not seen by Peano as a defect in his approach, but as a necessary feature of the best possible construction of geometry. An axiomatization is the result of a specific systematization of the properties of the primitive notions, but cannot be identified with an explicit definition.

Peano considers a second notion of purity, according to which the content of geometrical propositions is not determined by their informal wording, but by their deductive relations with the axioms. Accordingly, since the proof of Desargues's theorem requires the use of an axiom of solid geometry (and is, in fact, independent from linear geometry), for Peano this theorem belongs to solid geometry.

Geometry proper begins only once the axioms have been selected and formulated, at which point the theorems need to be demonstrated. And here again Peano's methodological principles determine the nature of this process of demonstration: it has to be regimented by logical laws and proceed exclusively by logical means. I claimed that the notions of purity that can be extracted from Peano's geometrical works and his understanding of a mathematical proof entails his endorsement of deductivism. It is then not surprising that Peano highlights the abstract character of this mathematical phase. The nature of the primitive notions is fundamental in the selection and formulation of the axioms, but once this basis is secured, their specific features become irrelevant. The consideration of a variety of interpretations of the primitive terms in the independence arguments that can be found both in *Principii di Geometria* [1889b] and 'Sui fondamenti della Geometria' [1894] bear witness of Peano's abstract understanding of the axioms.

References

Arana, A. (2008). Logical and Semantical Purity. *ProtoSociology*, *25*, pp. 36–48.

Arana, A.; Mancosu, P. (2012). On the Relationship between Plane and Solid Geometry. *The Review of Symbolic Logic*, *5*, pp. 294–353.

Avellone, M.; Brigaglia, A.; Zappulla, C. (2002). The Foundations of Projective Geometry in Italy from De Paolis to Pieri. *Archive for History of Exact Sciences*, *56*(5), pp. 363–425.

Bertran-San Millán, J. (2021a). Frege, Peano and the Construction of a Logical Calculus. *Logique et Analyse*, *253*, pp. 3–22.

Bertran-San Millán, J. (2021b). Frege, Peano and the Interplay between Logic and Mathematics. *Philosophia Scientiae*, *25*(1), pp. 15–34.

Bertran-San Millán, J. (2022). Peano's structuralism and the birth of formal languages. *Synthese*. Forthcoming.

Borga, M.; Freguglia, P.; Palladino, D. (1985). *I contributi fondazionali della Scuola di Peano*. Milano: Franco Angelli.

Bottazzini, U. (1985). Dall' Analisi Matematica al Calcolo Geometrico: Origini Delle Prime Richerche di Logica di Peano. *History and Philosophy of Logic*, *6*, pp. 25–52.

Bottazzini, U. (2001). I geometri italiani e il problema dei fondamenti (1889–1899). *Bollettino dell'Unione Matematica Italiana*, *4A*(2), pp. 281–329.

Bozzi, S. (2000). Vailati e la logica. In De Zan, M. (Ed.), *I mondi di carta di Giovanni Vailati*, Milano: Franco Angelli, pp. 88–111.

Brigaglia, A. (2016). Segre and the Foundations of Geometry: From Complex Projective Geometry to Dual Numbers. In Casnati *et al* [2016], pp. 265–288.

Cantù, P. (1999). *Giuseppe Veronese e i fondamenti della geometria.* Milano: Unicopli.

Casnati, G.; Conte, A.; Gatto, L.; Giacardi, L.; Marchisio, M.; Verra, A. (Eds.) (2016). *From Classical to Modern Algebraic Geometry. Corrado Segre's Mastership and Legacy.* Cham: Birkhäuser.

Conte, A.; Giacardi, L. (2016). Segre's University Courses and the Blossoming of the Italian School of Algebraic Geometry. In Casnati *et al* [2016], pp. 3–91.

Cremona, L. (1873). *Elementi di geometria projettiva.* Turin: G. B. Paravia.

De Paolis, R. (1880–81). Sui fondamenti della geometria proiettiva. *Atti della Regia Accademia dei Lincei*, *9*, pp. 489–503.

Detlefsen, M. (2008). Purity as an Ideal of Proof. In Mancosu [2008], pp. 179–197.

Detlefsen, M.; Arana, A. (2011). Purity of Methods. *Philosopher's Imprint*, *11*(2), pp. 1–20.

Dudman, V. H. (1971). Peano's review of Frege's *Grundgesetze*. *Southern Journal of Philosophy*, *9*, pp. 25–37.

Eder, G. (2021). Projective Duality and the Rise of Modern Logic. *The Bulletin of Symbolic Logic*, *27*(4), pp. 351–384.

Ewald, W. (Ed.) (1996). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics. 2 volumes.* Oxford: Oxford University Press.

Gandon, S. (2006). La réception des *Vorlesungen über neuere Geometrie* de Pasch par Peano. *Revue d'histoire des mathématiques*, *12*(2), pp. 249–290.

Genocchi, A. (1877). Sur un mémoire de Daviet de Foncenex et sur les géométries non euclidiennes. *Memorie della Reale Accademia delle Scienze di Torino*, *Series II, 29*, pp. 365–404.

Grassmann, H. (1844). *Die Lineale Ausdehnungslehre.* Leipzig: Otto Wigand.

Grassmann, H. (1862). *Die Ausdehnungslehre. Vollständig und in strenger Form begründet.* Berlin: Enslin.

Gray, J. (2007). *Worlds Out of Nothing. A Course in the History of Geometry in the 19th Century.* London: Springer.

Hallett, M. (2008). Reflections on the Purity of Method in Hilbert's *Grundlagen der Geometrie.* In Mancosu [2008], pp. 198–255.

Hallett, M.; Majer, U. (Eds.) (2004). *David Hilbert's Lectures on the Foundations of Geometry, 1891–1902.* Heidelberg: Springer.

Kennedy, H. C. (1972). The Origins of Modern Axiomatics: Pasch to Peano. *The American Mathematical Monthly*, *79*(2), pp. 133–136.

Klein, F. (1871). Über die sogennante Nicht-Euklidische Geometrie (Erster Aufsatz). *Mathematische Annalen*, *4*, pp. 573–625.

KLEIN, F. (1872). *Vergleichende Betrachtungen über neuere geometrische Forschungen.* Erlangen: A. Deichert.

KLEIN, F. (1873). Über die sogennante Nicht-Euklidische Geometrie (Zweiter Aufsatz). *Mathematische Annalen*, *6*, pp. 112–145.

KLEIN, F. (1889). Considerazioni comparative intorno a ricerche geometriche recenti. *Annali di Matematica pura e applicata*, *17*, pp. 307–343. Italian translation by G. Fano of Klein [1872].

KLEIN, F. (1890). Zur Nicht-Euklidische Geometrie. *Mathematische Annalen*, *37*, pp. 544–572.

KLEIN, F. (1911). Lecture VI: On the Mathematical Character of Space-Intuition, and the Relation of Pure Mathematics to the Applied Sciences. In *The Evanston Colloquium lectures on mathematics*, New York: McMillan, pp. 41–50.

LUCIANO, E.; ROERO, C. S. (2016). Corrado Segre and His Disciples: The Construction of an International Identity for the Italian School of Algebraic Geometry. In Casnati *et al* [2016], pp. 93–241.

MANARA, C. F.; SPOGLIANTI, M. (1977). La idea di iperspazio: Una dimenticata polemica tra G. Peano, C. Segre, e G. Veronese. *Memorie della Accademia Nazionale di Scienze, Lettere e Arti, di Modena*, *19*, pp. 109–129.

MANCOSU, P. (Ed.) (2008). *The Philosophy of Mathematical Practice.* Oxford: Oxford University Press.

MARCHISOTTO, E. A. (2011). Foundations of Geometry in the School of Peano. In SKOF, F. (Ed.), *Giuseppe Peano between Mathematics and Logic*, Milano: Springer, pp. 157–168.

MOORE, E. H. (1902). On the projective axioms of geometry. *Transactions of the American Mathematical Society*, *3*(1), pp. 142–158.

PADOA, A. (1901). Essai d'une théorie algébraique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque. In *Bibliothèque du Congrès international de philosophie, Paris 1900. Vol 3.* Paris: Armand Colin, pp. 309–365. Partial English translation by J. van Heijenoort in van Heijenoort [1967], pp. 118–123.

PASCH, M. (1882). *Vorlesungen über neuere Geometrie.* Leipzig: G. Teubner.

PEANO, G. (1888). *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva.* Turin: Fratelli Bocca. Partial reedition in Peano [1958], pp. 3–19. Partial English translation by H. C. Kennedy in Peano [1973], pp. 75–100. English translation in Peano [2000].

PEANO, G. (1889a). *Arithmetices principia nova methodo exposita.* Turin: Fratelli Bocca. Reedition in Peano [1958], pp. 20–55. English translation by H. C. Kennedy in Peano [1973], pp. 101–134. Partial English translation by J. van Heijenoort in van Heijenoort [1967], pp. 83–97.

PEANO, G. (1889b). *Principii di Geometria Logicamente Esposti.* Turin: Fratelli Bocca. Reedition in Peano [1958], pp. 56–91.

PEANO, G. (1891a). Osservazioni del Direttore sull'articolo precedente. *Rivista di matematica*, *1*, pp. 66–69.

PEANO, G. (1891b). Risposta. *Rivista di matematica*, *1*, pp. 156–159.

Peano, G. (1894). Sui fondamenti della Geometria. *Rivista di matematica*, *4*, pp. 51–90. Reedition in Peano [1959], pp. 115–157.

Peano, G. (1895). Recensione: Dr. Gottlob Frege, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet, Erster Band*, Jena, 1893. *Rivista di matematica*, *5*, pp. 122–128. Reedition in Peano [1958], pp. 189–195. English translation by V. Dudman in Dudman [1971], pp. 27–31.

Peano, G. (1897). *Formulaire de mathématiques*, vol. II, §1: Logique mathématique [F_2 §1]. Turin: Fratelli Bocca. Reedition in Peano [1958], pp. 218-281.

Peano, G. (1898). Analisi della teoria dei vettori. *Atti della Reale Accademia delle Scienze di Torino*, *33*, pp. 513–534.

Peano, G. (1900). Formules de Logique Mathématique. *Revue de Mathématiques*, *7*, pp. 1–41. Reedition in Peano [1958], pp. 304–361.

Peano, G. (1958). *Opere Scelte*, vol. II: Logica matematica, interlingua ed algebra della grammatica. Cassina, U. (Ed.), Roma: Cremonese.

Peano, G. (1959). *Opere Scelte*, vol. III: Geometria e fondamenti, meccanica razionale, varie. Cassina, U. (Ed.), Roma: Cremonese.

Peano, G. (1973). *Selected works of Giuseppe Peano*. Kennedy, H. C. (Ed.), London: Allen & Unwin.

Peano, G. (2000). *Geometric Calculus. According to the Ausdehnungslehre of H. Grassmann.* Boston: Birkhäuser.

Plücker, J. (1828–31). *Analytische geometische Entwicklungen (2 Bände)*. Leipzig: Teubner.

Plücker, J. (1868). *Neue Geometrie des Raumes*. Leipzig: Teubner.

Rizza, D. (2009). Abstraction and Intuition in Peano's Axiomatizations of Geometry. *History and Philosophy of Logic*, *30*(4), pp. 349–368.

Schlimm, D. (2010). Pasch's Philosophy of Mathematics. *The Review of Symbolic Logic*, *3*(1), pp. 93–118.

Segre, C. (1883). Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni. *Memoria della Reale Academia delle Science di Torino*, *36*, pp. 3–86. Reedition in Segre [1961], pp. 25–126.

Segre, C. (1891a). Su alcuni indirizzi nelle investigazioni geometriche. *Rivista di matematica*, *1*, pp. 42–66. Reedition in Segre [1963], pp. 387–412. English translation by J. W. Young in Segre [1904].

Segre, C. (1891b). Un dichiarazione. *Rivista di matematica*, *1*, pp. 154–156.

Segre, C. (1904). On some tendencies in geometric investigations. *Bulletin of the American Mathematical Society*, *10*(9), pp. 442–468.

Segre, C. (1961). *Opere*, vol. III. Roma: Cremonese.

Segre, C. (1963). *Opere*, vol. IV. Roma: Cremonese.

van Heijenoort, J. (Ed.) (1967). *From Frege to Gödel, a Source Book in Mathematical Thought*. Cambridge: Harvard University Press.

Veronese, G. (1891). *Fondamenti di geometria a più dimensioni e a più specie di unità rettilinee esposti in forma elementare*. Padova: Tipografia del Seminario Padova.

von Staudt, K. G. C. (1847). *Geometrie der Lage*. Nürnberg: F. Korn.

von Staudt, K. G. C. (1889). *Geometria di posizione*. Turin: Fratelli Bocca. Italian translation by M. Pieri of von Staudt [1847].

Whitehead, A. N. (1906). *The Axioms of Projective Geometry.* London: Cambridge University Press.