PEANO'S STRUCTURALISM AND THE BIRTH OF FORMAL LANGUAGES

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Abstract. Recent historical studies have investigated the first proponents of methodological structuralism in late nineteenth-century mathematics. In this paper, I shall attempt to answer the question of whether Peano can be counted amongst the early structuralists. I shall focus on Peano's understanding of the primitive notions and axioms of geometry and arithmetic. First, I shall argue that the undefinability of the primitive notions of geometry and arithmetic led Peano to the study of the relational features of the systems of objects that compose these theories. Second, I shall claim that, in the context of independence arguments, Peano developed a schematic understanding of the axioms which, despite diverging in some respects from Dedekind's construction of arithmetic, should be considered structuralist. From this stance I shall argue that this schematic understanding of the axioms anticipates the basic components of a formal language.

1. Introduction

The revival of structuralism in the philosophy of mathematics has recently motivated an interest in the history of structuralism and, in particular, in the first proponents of this approach in late nineteenth-century mathematics.[1] Dedekind often plays a prominent role as one of the early advocates of structuralism. Despite the similarities between Peano's construction of arithmetic and Dedekind's, the possibility that Peano can be counted among the early structuralist mathematicians has been seldom considered in historical studies.[2]

Leaving aside the fact that most of Peano's writings have not been translated into English, Peano's style of writing might have played a role in this regard. Philosophical discussion is sparse in Peano's works. Accordingly, it is particularly difficult to offer a rational evaluation of Peano's philosophical views. His philosophy – and, specifically, his philosophy of mathematics – has to be extracted, not from explicit discussions but from his presentation

[1]Regarding the attention that the prehistory of structuralism has received in recent years, see for instance [Reck; Schiemer, 2020a].

[2]To the best of my knowledge, only Cantù [2021] has taken this possibility into consideration. Cantù characterises Peano's approach as structuralist algebraism and focusses on Peano's notions of interpretation and meaning, his use of abstraction, and the metamathematical use of definitions. Since I only consider these topics in passing (see Footnote 6), this paper can be seen as complementary to Cantù's. See also [Rizza, 2009, pp. 365–367]. Rizza [2009] focusses on the apparent tension between Peano's empiricism and his abstract approach to geometry; only in a final section does Rizza briefly consider the philosophical significance (partly in connection to structuralism) of Peano's construction of geometry, but he neither articulates any specific form of structuralism that could fit Peano's methodology nor studies Peano's work under that light.

of mathematical theories, methodology, and remarks regarding his results. In this regard, the question of whether Peano was a structuralist can only be answered from the methodological or mathematical point of view.[3] Peano never articulated a metaphysical form of structuralism; in other words, he never addressed philosophically the question of the nature of the abstract structures and relations that mathematical theories are (purportedly) about. In this paper, by studying Peano's axiomatisation of geometry and arithmetic, I shall attempt to answer the question: was Peano a methodological structuralist?

I shall assume that methodological structuralism involves at least two elements.[4] First, methodological structuralism requires the study of the relational features of the systems of objects that compose mathematical theories, that is, an emphasis on structural elements rather than on the nature of the objects themselves. Second, it assumes that there are multiple systems that exhibit these relational features. That said, as Reck and Schiemer suggest, a characterisation of methodological structuralism can be a matter of "family resemblance" [2020b, p. 10]. Other features can be associated with this form of structuralism. Specifically, to the second condition it could be added that methodological structuralism involves the study and systematic comparison of the systems that share a particular collection of relational features; this study is typically done in terms of morphisms, and can lead to the identification of isomorphic systems (see [Reck; Schiemer, 2020b, pp. 9–10]).

Dedekind's construction of the system of natural numbers in *Was sind und was sollen die Zahlen?* [1888] is perhaps the best known example of methodological structuralism. On the one hand, in the definition of a simply infinite system – which encapsulates the properties of the system of natural numbers – Dedekind states that only the relations established in the four conditions included in the definition are considered, and the special character of the natural numbers is neglected [1888, Sects. 71–73, pp. 359–360]/[Ewald, 1996, pp. 808–809].[5] On the other hand, Dedekind explicitly acknowledges multiple systems that satisfy the definition of a simply infinite system. In fact,

[3]The term 'methodological structuralism' was coined as a means to differentiate those structuralist views that do not fit in the philosophically involved approach of metaphysical structuralism. See [Awodey, 1996] and [Reck, 2003, pp. 370–374]. For the purposes of this paper, it is enough to state that *metaphysical* or *philosophical structuralism* aims at clarifying the nature of mathematical structures, while *methodological* or *mathematical structuralism* corresponds to a methodology, according to which the object of mathematical theories are mainly structures instead of the objects that compose them. Methodological structuralism does not involve the kind of metaphysical and epistemological discourse typical for metaphysical structuralism. See below for a more precise characterisation of a minimal methodological structuralism. On a taxonomy of mathematical structuralism and, specifically, on the distinction between methodological and metaphysical structuralism, see [Reck; Price, 2000].

[4]In this twofold understanding of a minimal methodological structuralism I follow [Schlimm, 2020, pp. 89–91]. I find his arguments against the triviality of such a characterisation of methodological structuralism convincing. See also [Reck, 2003, pp. 371–372] and [Reck; Schiemer, 2020b, pp. 4–11].

[5]Unless a reference to an English translation is included after a slash, all quotations of the sources are translated by the author. Page numbers refer to the most recent edition of the source or translation listed in the Bibliography.

he studies the class of simply infinite systems and demonstrates theorems which establish the isomorphism between the elements of that class [1888, Sects. 132–134, pp. 376–378]/[Ewald, 1996, pp. 821–823].

In this paper, I shall focus on Peano's interpretation of the primitive notions of geometry and arithmetic, and of the axioms of these two theories.[6] In this particular context, I shall argue that, although Peano did not fully exploit the implications of the second requirement of methodological structuralism, he anticipated key aspects of a structural understanding of mathematical theories.

This paper is in four parts. After this introduction, in the second section, I shall characterise Peano's particular use of the logical formalism he devised. With methodological as well as pedagogical motivations in mind, Peano offered presentations of mathematical theories – often in an axiomatic way – by means of a combination of logical and mathematical symbolism, which I shall refer to as 'symbolisation'.[7] In the third section, I shall explain how Peano conceived the basic notions of arithmetic and geometry and how these notions were involved in the axiomatisation of these theories. Specifically, I shall claim that, despite Peano's empiricist position and his requirement for non-abstract foundations, the indefinability of the primitive notions led him to formulate the axioms of geometry and arithmetic in such a way that they express the relational features of the systems of objects that compose these theories. In the fourth section, as a means to contextualise Peano's structuralist methodology, I shall compare Peano and Dedekind's constructions of arithmetic. I shall attempt to clarify the similarities and differences of these two approaches and argue that, although they can both be considered structuralist, Dedekind's definition of a simply infinite system and Peano's axiomatisation of arithmetic should be distinguished. Finally, in the fifth section, I shall consider one particular context in which Peano developed an alternative understanding of the axioms of geometry. I shall demonstrate how Peano used his symbolisation of geometry and adopted – in some respects – a model-theoretic point of view to show that the axioms of these theories are mutually independent. By so doing, as I shall claim, he effectively deployed most of the tools required to build a formal language.

2. The symbolisation of mathematical theories

Peano's first contribution to logic is a chapter in *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann* [1888], which contains a presentation of Boolean logic. In *Arithmetices principia nova methodo exposita* [1889a] (hereinafter, *Arithmetices principia*) he remodels his mathematical logic with – among other things – a new symbolism and an adequate way of representing quantification. Peano's purpose in 1889 is not, however, the development of a complete logical system, but the creation of a tool adequate for the rigorous, precise and clear presentation of arithmetic. In fact, in

[6]This leaves out of consideration relevant issues that, for reasons of space, I shall not deal with in this paper, such as Peano's definitions by abstraction and his notion of implicit definition. Recent publications have already discussed these topics; see [Mancosu, 2018] and [Cantù, 2021]. See also [Borga *et al*, 1985].

[7]On the importance of pedagogical matters in the Peano school, see [Marchisotto; Millán Gasca, 2021].

Arithmetices principia, Peano does not even axiomatise either his calculus of classes or his sentential calculus; he just offers a list of the logical laws that the derivations of arithmetical theorems require. Using the logical apparatus he has introduced, Peano states that "every proposition assumes the form and precision equations enjoy in algebra, and from propositions so written others may be deduced, by a process which resembles the solution of algebraic equations" [1889a, p. 21]/[1973, p. 102]. Also in 1889, in *Principii di Geometria logicamente esposti* (hereinafter, *Principii di Geometria*) [1889b], Peano uses logic as a tool for the rigorous construction of projective geometry. In *Notations de logique mathématique*, he describes this combination of logic and mathematical theories:

> Any theory can be reduced to symbols, for every spoken language, and every piece of writing [*écriture*], is a symbolism, or a series of signs that represent ideas. In order to apply the signs we have explained, we can take the propositions of the theory in question, written in ordinary language, and replace the word *is* with the signs ϵ, $=$, $\supset$, whatever the case may be, and [put] instead of *and*, *or*, ... the signs $\cap$, $\cup$, ...; and that *cum granu salis*, because we saw for instance that, depending on the position, the conjunction *and* is represented by means of $\cap$ or $\cup$.
>
> After this first transformation, the propositions are expressed in a few words, linked by the logical signs $\cap$, $\cup$, $=$, $\supset$, etc.; and if it has been well done, the words that remain are devoid of any grammatical form; for all the relations of grammar are expressed by means of logical signs. These words represent the proper ideas of the theory being studied. Then the ideas represented by these words are analysed, the composed ideas are decomposed into the simple parts, and only then, after a long series of reductions and transformations, does one obtain a small group of words, which can be considered the "minimum", by means of which, combined with the signs of logic, all the ideas and propositions of the science under study can be expressed. [Peano, 1894b, p. 164]

Peano's approach can be presented in a more precise way. I shall characterise the reduction Peano alludes to as a symbolisation.[8] Let T be a theory, that is, a collection of statements expressed in a language L^* and closed under logical consequence.[9] Paradigmatically, T is a mathematical theory. The language L^* contains a set C of non-logical constants (i.e., individual constants, function symbols, and predicate and relation symbols) that are

[8] I do not claim to use this term in a standard way. I have chosen it to refer to a particular application of logic Peano devised, although he did not use 'symbolisation' with this specific meaning in mind. The term 'formalisation' could be used to refer to what I call 'symbolisation'; nevertheless, the former term shall be employed from Section 5.2 onwards with a specific sense.

[9] Although Peano – to a certain extent – developed the tools required to construct a deductive calculus with his mathematical logic, he never offered a precise characterisation of the notion of logical consequence. For the purposes of this paper, it shall be enough to emphasise Peano's methodological principle that a theory should be axiomatised and to retain the intuitive idea held by Peano that the theorems of a theory are obtained from the axioms by logical means. See Section 5.1. On the development of Peano's logical calculus, see [Bertran-San Millán, 2021].

used to refer to the entities that compose the meaning of the statements of T. Let L be a logical language and let L' be an extension of L with C, that is, let $L' = L \cup C$. By *symbolisation* of T I understand, firstly, the reformulation of the statements of T in the language L' in such a way that (i) the specific meaning of the statements of T is essentially maintained (by means of the use of the symbols in C and the symbolic representation of the logical form of the statements of T); and (ii) all logical relations expressed in the statements of T are formulated using the logical constants and variables of L'.

A symbolisation as Peano conceived it typically also involves, secondly, the axiomatisation of the theory T in terms of L'; thirdly, the definition in terms of L' of derived notions involved in the statements of T; and, lastly, the use of a logical calculus formulated in L for conducting the proofs of the reformulated theorems of T.

Accordingly, a symbolisation does not consist in a mere rewriting. It is the result of an analysis of the notions involved in the statements of T, in the sense that it includes a selection of primitive notions and the organisation of their basic properties in an axiomatisation. Moreover, a symbolisation is also a means to rigorously express the meaning conveyed by the laws of T.

It is often the case that the statements and proofs of a mathematical theory are formulated in a combination of natural language and mathematical symbols. Since, for Peano, natural language is prone to ambiguities and inaccuracies, one of the purposes of symbolisation is to avoid the use of natural language in the reformulation of the statements of a theory and also in its proofs (see [Peano, 1889a, pp. 21–23]/[1973, pp. 101–103]).

Peano's best known example of symbolisation is his presentation of arithmetic. In *Arithmetices principia*, he specifies the components of the logical language L and the set of non-logical constants $C = \{=, \mathrm{N}, 1, +\}$ [1889a, p. 28]/[1973, pp. 103–104].[10] This symbolisation includes, as is well known, the first axiomatisation of arithmetic and the definition of all the derived notions of arithmetic, e.g., the subtraction operation or the relation D of being a divisor. Moreover, a substantial part of Peano's [1889a] symbolisation of arithmetic consists in the reformulation of theorems and the proof of some of them by logical means. For instance, Peano formulates the theorem that the sum of two multiples of a number is in turn a multiple of that number as follows [1889a, p. 44]/[1973, p. 123]:

$$a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\mathrm{D}\, a \,.\, c \,\mathrm{D}\, b : \mathrel{\reflectbox{c}} .\, c \,\mathrm{D}\, a + b. \tag{27}$$

The effort at offering a formulation of arithmetic that fulfils Peano's standards of clarity, precision, and rigour culminated in the successive editions

[10]The elements of C are the primitive terms of arithmetic and are left undefined. On Peano's understanding of the primitive terms, see the following section. In *Arithmetices principia*, by means of definitions the set C is enlarged with the collection $\{>, <, -, \times, /, \mathrm{N}, \mathrm{R}, \mathrm{Q}, \mathrm{Np}, \mathrm{M}, \reflectbox{M}, \mathrm{T}, \mathrm{D}, \reflectbox{D}, \pi\}$ and the set of numerals $\{2, 3, \dots\}$.

Peano includes the equality symbol in the list of arithmetical symbols. From 1891 on, he takes it as a genuine logical symbol. See [Peano, 1891d, p. 84], where Peano does not include the axioms that express the properties of the equality relation in the list of arithmetical axioms.

of the *Formulaire de mathématiques*, published between 1894 and 1908.[11] In the first years of the 1890s, before he focussed on the *Formulaire*, Peano published symbolisations of geometry [1889b; 1894a], analysis [1890b], number theory [1891d; 1892b] and even Euclid's *Elements* [1890a; 1891a; 1892a].

For Peano, even though he refers to a symbolisation as a reduction, the symbolisation of mathematical theories has no logicist purpose. It aims at precision and rigour in the formulation of the laws of a theory, its definitions and proofs. An instrumental element in this regard is the elimination of natural language in scientific and, specifically, in mathematical contexts. Peano defends symbolisation as a means to precisely characterise the notions of mathematical theories and, in particular, to rigorously express all logical relations involved in a definition [1889b, pp. 56–58].

3. The primitive notions of arithmetic and geometry

Underlying the symbolisation of geometry and arithmetic Peano performs in his early works on logic and later in successive editions of the *Formulaire de mathématiques*, there is the conviction that mathematical terms have established and recognisable meanings, which are accessible to the mathematician.[12]

3.1. Concrete and abstract foundations of geometry and arithmetic.

In 'Sui fondamenti della Geometria', Peano selects the concepts of point and the relation "lie between" (established between three points) as the primitive notions of geometry, and states that these are "very simple ideas, common to all men" [1894a, p. 116]. In *Principii di Geometria*, Peano already devises a symbolic representation of these notions: '**1**' corresponds to the class of points and '$c \epsilon ab$' to the circumstance that a point c lies between points a and b (i.e., c lies in the segment determined by a and b).[13] As primitive notions, the basic concepts of geometry are undefined.

For Peano, these basic concepts require a secure ground. In 'Sui fondamenti della Geometria', he states that they "must be obtained by experience [*esperienza*]" [1894a, p. 119] and that their properties are acquired by "the most elementary observations" [1894a, p. 119].

[11]On Peano's collective project of a *Formulaire de mathématiques* – or, as he would call it in later editions, *Formulario mathematico* – see [Borga *et al*, 1985, pp. 163–170] and [Lolli, 2011].

[12]In other words, Peano assumes in his works – at least at an initial stage – that mathematical terms and, specifically, the primitive terms of mathematical theories, are *substantive*. Mathematical theories determine an ontology composed of the entities over which the statements of the theory quantify. Mathematical terms are substantive when they refer to the entities of this ontology, and thus their sense can be reconstructed (either by definition or by any other means, if they are primitive). My use of the notion of substantive term is based on [Klev, 2011].

[13]The term '$c \epsilon ab$' is, strictly speaking, complex. On the one hand, 'ϵ' is the membership relation symbol and belongs to the language of the calculus of classes. On the other, the term 'ab' refers to the segment determined by a and b (the class of points that lie between a and b), i.e., to the result of applying the segment formation function to points a and b. As we shall see in what follows and in Section 5.3, on some occasions Peano refers to '$c \epsilon ab$' as if it expresses that a, b and c stand in a ternary relation.

Peano thus shares Pasch's idea that the primitive notions of geometry cannot be defined and must be acquired empirically [Pasch, 1882, p. iv, p. 16].[14] Peano's view on the need for an empirical foundation of geometry was not unusual among Italian mathematicians either. Some members of the Italian school of algebraic geometry – such as Veronese or Enriques – also argued for empiricism. For instance, Veronese – with whom Peano held a bitter polemic – claimed that geometry is an experimental science and that the establishment of its axioms requires external observation [1891, pp. viii, ix].[15] In contrast, other members of the Italian school defended an abstract foundation of geometry, according to which the nature of the basic geometrical concepts – e.g., that which makes a point recognisable as a point – is irrelevant (see, for instance, [Segre, 1883, p. 39]). In Peano's polemic with Segre on the requirement of rigour in geometry, the former criticises a purely abstract conception which begins in "hypotheses contrary to experience, or [in] hypotheses which cannot be verified by experience" [Peano, 1891c, p. 67].

In his critique of Segre's construction of hyperspace geometry, Peano does not question an abstract development of the discipline, but its lack of empirical foundation. In other words, he requires the primitive concepts of geometry to be grounded on experience.

Concerning the primitive terms of arithmetic, in *Arithmetices principia*, Peano says little more than that "[t]he sign N means *number* (*positive integer*); 1 means *unity*; $a + 1$ means the *successor of* a, or a *plus* 1" [1889a, p. 34]/[1973, p. 113]. It is assumed that these symbols refer to the fundamental notions of arithmetic and thus are the building blocks of the symbolisation of arithmetic. That said, Peano avoids any elucidation, in the Fregean sense, of the nature of these notions; that is, any clarification that makes the sense of their corresponding terms accessible.[16] In 'Sul concetto di numero' [1891d], he poses the question of whether the primitive concepts of arithmetic can be defined. His answer is twofold. From a practical or pedagogical point of view, Peano states that "it is not convenient in teaching to give any definition of number, this idea being very clear to the students, and each definition having the effect of confusing it" [Peano, 1891d, p. 84]. From the theoretical point of view, Peano claims that using only logical symbols, "the number cannot be defined, since it is evident that however those words [corresponding to logical constants] are combined, it will never be possible to have an expression equivalent to a number" [Peano, 1891d, p. 85].

In sum, for Peano, the primitive notions of geometry and arithmetic must be known and recognisable notions. In fact, the basic concepts of geometry

[14]In *Vorlesungen über neuere Geometrie*, Pasch distinguishes between basic and stem concepts [1882, p. 74]. The former are undefined, acquired empirically, and correspond to the philosophical foundation of the theory. In Pasch's [1882] treatise, the stem concepts of projective geometry are derived from basic propositions and concepts. On Pasch's empiricism, see [Schlimm, 2010]. On Pasch's influence in Peano's early presentations of geometry, see [Borga *et al*, 1985, pp. 52–54]. On an alternative interpretation of the relation between Peano's requirement for the empiric foundation of the axioms of geometry and Pasch's empiricism, see [Gandon, 2006, pp. 272–280].

[15]See [Avellone *et al*, 2002] for an informed overview of empiricism in the Italian school. On the polemic between Veronese and Peano, see [Borga *et al*, 1985, pp. 244-250].

[16]On Frege's notion of elucidation (*Erläuterung*), see [Frege, 1906, p. 288]/[1984, pp. 300–301].

are founded in direct observation. Hence, the primitive terms ought to be substantive in order to be chosen. This does not mean that it is fully determined whether a specific group of concepts is the collection of primitive notions. Peano acknowledges that the choice of primitives always bears some arbitrariness – at least, as long as the combination of derived and primitive notions is the same (see [Peano, 1889b, p. 78] and [Peano, 1891b, p. 104, fn. 2]). Whatever the particular selection of primitive notions may be, these basic concepts cannot be defined. In fact, as Peano states in 'Sul concetto di numero', they cannot even be elucidated if his scientific standards are to be met [1891d, p. 84]. If no definition or elucidation of the fundamental concepts of geometry and arithmetic is possible, these mathematical theories have to be constructed from axioms – which in the case of arithmetic Peano calls 'primitive propositions'. This is exactly how most of Peano's symbolisations start: once the primitive notions are identified, the axioms are lain down using the formal resources provided by mathematical logic.

3.2. **Primitive notions and axioms.** The axioms of geometry are formulated on the basis of empirical facts that explain our access to the primitive notion of point and the primitive relation "to lie between". In 'Sui fondamenti della Geometria', Peano states the following:

> [I]t will be necessary to determine the properties of the undefined entity p [point], and of the relation $c \varepsilon ab$ [c lies between a and b], by means of axioms or postulates. The most elementary observation shows us a long series of properties of these entities; we just have to collect these common cognitions, order them, and enunciate as postulates only those that cannot be deduced from simpler ones. [Peano, 1894a, p. 55]

The primitive notions cannot be defined, but their properties can be stated in the axioms and, further, in the theorems that are derived from them.

Something similar can be said of the axioms of arithmetic. In 'Sul concetto di numero', Peano presents the axioms of arithmetic as "[t]he propositions which state the most simple properties of the [positive] integers" [1891d, p. 85]. Thus, the axioms of arithmetic represent fundamental facts about notions well known to the mathematician, who is presupposed to have certain intuitions about the notions of natural number, the first element of the numerical series, and the successor operation.

Peano's attempt to deal rigorously with the foundations of geometry and arithmetic and, specifically, to present these theories axiomatically, has clear and explicitly attested precedents. Pasch's *Vorlesungen über neuere Geometrie* [1882] is a groundbreaking work on the foundations of geometry. Pasch advocates for the methodological requirements of the explicit establishment of primitive notions and axioms, and the sharp distinction between theorems and axioms [1882, pp. 4–5].[17] In *Principii di Geometria*, Peano acknowledges

[17] In *Vorlesungen über neuere Geometrie*, Pasch states the following:

> Mathematics establishes relations between the mathematical concepts which are supposed to correspond to the facts of experience. Nevertheless, the vast majority [of these concepts] is not directly borrowed from experience but is "proven". Apart from the definitions of the derived concepts, the knowledge necessary for the demonstration is itself part of

that most of his axioms of linear geometry correspond to Pasch's axioms [1889b, pp. 84–85] (see also [Peano, 1894a, pp. 119–120]).[18] In arithmetic, Peano's immediate predecessor is Hermann Grassmann. Even though it is intended as a textbook for teachers, in *Lehrbuch der Arithmetik für höhere Lehranstalten* [1861], Grassmann attempts to lay down the foundations of arithmetic in a rigorous way but falls short of a properly axiomatic presentation.[19] Grassmann's recursive definitions of arithmetical operations were very influential for Peano, who gives due recognition to Grassmann's [1861] treatise in the Preface of *Arithmetices principia* [1889a, p. 22]/[1973, p. 103]. In contrast to Pasch and Grassmann, for Peano a rigorous presentation of geometry and arithmetic requires not only their axiomatisation, but also their complete symbolisation. This ensures a clear, precise presentation, which is also free of ambiguities. In *Principii di Geometria*, Peano analyses the content of Pasch's first axiom, namely, "One straight line, and one only, can always be drawn between two points" [Pasch, 1882, p. 5], and provides four possible symbolisations of it that dissolve the ambiguities of Pasch's formulation [Peano, 1889b, pp. 84–85].[20]

It has already been said that Peano presents in *Arithmetices principia* the first axiomatisation of arithmetic. In the third part of the second volume of *Formulaire de mathématiques*, Peano provides a refined symbolisation of the axioms of arithmetic:[21]

Primitive propositions

> the relations to be established. After the elimination of the propositions based on proofs, the *theorems* [*Lehrsätze*], a group of propositions remains, from which all the rest can be deduced, the *basic propositions* [*Grundsätze*]; these are based directly on observations [...]. [Pasch, 1882, p. 17]

See Footnote 14 and [Schlimm, 2010].

[18]On a comparison between Pasch [1882] and Peano's [1889b] axioms, see [Borga *et al*, 1985, pp. 206–211].

[19]On the significance of Grassmann's work for the axiomatic method, see [Radu, 2011]. On Grassmann's definitions by recursion and his influence on Peano, see [von Plato, 2017, pp. 40–57]. See also [Cantù, 2020].

[20]Peano's four possible symbolisations of Pasch's aforementioned axiom are the following:

$$a, b \,\epsilon\, \mathbf{1} \,.\, \supset .\, ab \,\epsilon\, \mathrm{K}\mathbf{1},$$
$$a, b \,\epsilon\, \mathbf{1} \,.\, a \mathbin{-\!\!=} b \colon \supset .\, ab \,\epsilon\, \mathrm{K}\mathbf{1},$$
$$a, b, c, d \,\epsilon\, \mathbf{1} \,.\, a = b \,.\, c = d \colon \supset .\, ac = bd,$$
$$a, b, c, d \,\epsilon\, \mathbf{1} \therefore a = b \,.\, c = d \colon \cup \colon a = d \,.\, b = c :: \supset .\, ac = bd,$$

where '$\mathrm{K}\mathbf{1}$' stands for the class of classes of points and thus '$ab \,\epsilon\, \mathrm{K}\mathbf{1}$' means that ab is a class of points or a geometrical figure [1889b, p. 59].

After the suggestion of the possible symbolisations of Pasch's axiom, Peano motivates the use of a symbolisation thus:

> We see from this brief discussion how difficult it is in such delicate matters, even for an accurate writer [Pasch], to avoid any danger of ambiguity, if one proceeds with natural language. In order to overcome this difficulty, it is necessary to analyse each proposition, and to fix completely the meaning [*valore*] of the terms we use. In doing so, one necessarily arrives either at the logical notations, which I use here, or at an equivalent system. [Peano, 1889b, p. 85]

[21]Note that $a+$ is the successor of a, for any individual a.

$\cdot 1\ \ 0 \,\varepsilon\, \mathrm{N}_0$	Pp
$\cdot 2\ \ a \,\varepsilon\, \mathrm{N}_0 \,.\, \supset .\, a+ \,\varepsilon\, \mathrm{N}_0$	Pp
$\cdot 3\ \ a, b \,\varepsilon\, \mathrm{N}_0 \,.\, a+ = b+ \,.\, \supset .\, a = b$	Pp
$\cdot 4\ \ a \,\varepsilon\, \mathrm{N}_0 \,.\, \supset .\, a+ \,-\!\!= 0$	Pp
$\cdot 5\ \ s \,\varepsilon\, \mathrm{Cls} \,.\, 0 \,\varepsilon\, s : x \,\varepsilon\, s \,.\, \supset_x .\, x+ \,\varepsilon\, s : \supset .\, \mathrm{N}_0 \supset s$	Pp

[Peano, 1899, p. 29]

This axiomatisation is essentially equivalent to that given in 'Sul concetto di numero' [1891d, p. 84]. The primitive propositions are understood as expressing, in a rigorous and precise way, the fundamental properties of the primitive notions of arithmetic. However, these fundamental properties do not correspond to the intrinsic features of the primitive notions, namely, those properties that identify the basic concepts as a specific class (the class of natural numbers), individual (the number 0), and operation (the successor operation). In other words, Peano's axioms of arithmetic establish general conditions that determine the relations between a class, an object, and an operation, but cannot be used to single out a specific system. In the axioms, the number 0 is not characterised as an object with certain specific properties, but as the first element of the numeric series; only its relational properties (e.g., the fact that it is not the successor of any number, as the primitive proposition ·4 expresses) are established. If we consider the class of odd numbers, the number 1, and the operation +2, then by appealing only to the axioms we would have no way to distinguish 1 and 0 as the initial elements of the class of odd numbers and the class of natural numbers, respectively. The axioms solely characterise a class of structured systems to which the systems of the natural numbers and of the odd numbers belong.[22] As we shall see in the next section, there is no acknowledgement in *Arithmetices principia* of the fact that the axioms do not characterise a unique system of objects. From 1891 onwards, Peano explicitly advocates for a structuralist understanding of the axioms.[23]

The axioms of geometry express relations between individuals and collections of individuals that are shared by a class of systems, but do not properly characterise a single system of entities of that class. As we shall see in Section 5.3, Peano's presentation of the axioms of linear geometry in *Principii di Geometria* includes the consideration of systems which satisfy them [1889b, pp. 83–89]. In this sense, the axioms are understood in an abstract way.[24] Peano confirms this idea in the notes that follow the symbolisation of elementary geometry in *Principii di Geometria*. He expresses himself thus:

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[22] This reading of Peano's axioms essentially coincides with Russell's. In *Principles of Mathematics*, Russell states that Peano neither succeeds in indicating any constant meaning of 0, number, and succession, nor in showing that any constant meaning is possible [1903, §122, p. 126].

[23] This does not imply that Peano was not aware of such an understanding already in 1889a. Not only had Peano read Dedekind's *Was sind und was sollen die Zahlen?* [1888] by 1889; in *Principii di Geometria* Peano also considers several interpretations of the axioms of linear geometry which satisfy some or all of them.

[24] Similar accounts can be found in recent historical studies. Rizza summarises Peano's axiomatic construction of geometry as follows:

> [T]he need to systematically organize spatial intuition around certain fundamental concepts can give rise to the concept of a formal structure as a type of organization of a given intuitive content. The choice of

> [T]here is a category of entities, called points. These entities are not defined. Moreover, given three points, a relationship between them is considered, expressed by the expression [*scrittura*] $c \epsilon ab$, which likewise is not defined. The reader can understand [*intendere*] by the sign **1** any category of entities, and by $c \epsilon ab$ any relationship between three entities of that category; all the definitions that follow (§2) will always have a value, and all the propositions of §3 will be founded [*sussisteranno*]. [Peano, 1889b, p. 77]

Peano states that there is no predetermined way to interpret the primitive terms '**1**' and '$c \epsilon ab$', and thus their meaning does not matter. This can be understood as the adoption of an abstract perspective, according to which the intrinsic nature of the primitive notions is not captured by the axioms. In sum, for Peano, the empirical foundation of the primitive notions of geometry is not incompatible with an axiomatisation that involves an abstract understanding of the terms that correspond to the primitive notions.[25]

The selection of the primitive notions of geometry has to be informed by experience. Similarly, the establishment of the primitive notions of arithmetic relies on intuitions regarding the nature of the number. The selection of the basic concepts of geometry and arithmetic thus belongs to a pre-mathematical phase in Peano's construction of these theories (see [Peano, 1891c, p. 67]). The axioms constitute a particular analysis of the properties of the primitive notions, and their selection and symbolisation also belongs to this first phase. However, there is a second phase, properly mathematical, where the axioms are used to develop the theory in question. In fact, as we have seen, the axioms do not determine a single system that satisfies them. Accordingly, in the process of development of geometry and arithmetic, the primitive terms are severed from their original meaning. To this second phase belong the derivation of theorems and the study of the deductive relations between the axioms. The derivation of theorems does not need to rely on any intuition of space, of the numeric series, or direct observation. As we shall see in sections 5.1 and 5.3, Peano's strategy of ignoring the original meaning of the primitive terms is essential in his conception of a mathematical proof and in the investigation of metatheoretical questions such as the independence of the axioms.

All in all, this study of Peano's axiomatisation of arithmetic and geometry yields the conclusion that these systems of axioms are not intended to characterise the specific nature of the primitive notions – which remain undefined. Peano's resolute commitment to the axiomatic method in arithmetic and geometry is not at odds with his view that the primitive notions are specific entities and the use of the corresponding terms indicates that their meaning is accessible to a mathematician; a characterisation of these notions by means

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> fundamental concepts and the articulation of geometry on their basis is carried out through the axiomatic method. [Rizza, 2009, p. 366]

[25]On the compatibility between the abstract understanding of the primitive terms of geometry and the requirement of an empirical foundation, see [Bertran-San Millán, 2022] and [Rizza, 2009].

of axioms cannot be identified with an explicit definition.[26] Only the properties and relations specified in the axioms of arithmetic and geometry are considered, and this shows that Peano's symbolisation of these two theories fulfils the first requirement of a minimal methodological structuralism (see page 2).

4. Peano and Dedekind on the construction of arithmetic

In the investigation of Peano's endorsement of methodological structuralism, Dedekind's work is a relevant landmark.[27] Not only is Dedekind one of the most studied examples of early structuralist mathematicians; his works were also well known by Peano, who quoted them several times. In this section, I shall attempt to clarify the similarities and differences between Peano and Dedekind's structuralist accounts. For this purpose, I shall focus on their construction of arithmetic.

4.1. **Peano's structuralist understanding of the axioms.** Although by 1889 Peano had already formulated an axiomatisation of arithmetic, an explicit structural interpretation of the axioms can only be found in the works on arithmetic he published after 1889. In the Preface to *Arithmetices principia* [1889a, p. 22]/[1973, p. 103], he refers to *Was sind und was sollen die Zahlen* [1888]. However, in that work, Peano does not provide any remark on his understanding of the axioms.

That said, soon after 1889, Peano made it clear that he was aware that his strategy, which avoids defining the primitive notions and offers instead an axiomatisation of the theory, is different from Dedekind's, whose construction of arithmetic starts from the description of the primitive logical concepts and the definition of the primitive notions of arithmetic. In 'Sul concetto di numero', Peano concedes that the two strategies coincide in their results:

> There is an apparent contradiction between the foregoing [Peano [1891d] axiomatisation of arithmetic] and what Dedekind says,

[26]Peano's conception of the construction of a mathematical theory and, specifically, his dismissal of the elucidation of the primitive notions and his preference for axiomatisation, is close to Hilbert's early conception. In a letter to Frege dated December 29, 1899, Hilbert expresses doubts about the definition of the notion of point, which are very similar to Peano's reasons for discarding a definition of the basic notions of arithmetic:

> If one is looking for other definitions of a 'point', e.g., through paraphrase in terms of extensionless, etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find because there is nothing there; and everything gets lost and becomes vague and tangled and degenerates into a game of hide-and-seek. If you prefer to call my axioms characteristic marks of the concepts which are given and hence contained in the 'explanations', I would have no objection at all [...]. [Frege, 1976, p. 66]/[1980, p. 39]

In Sieg's [2014] terms, it could be said that Peano's understanding of the primitive terms can be framed within existential structuralism. On Hilbert's early conception of the primitive terms of mathematical theories, see also [Klev, 2011, pp. 671–673]. On the similarities between Peano's and Hilbert's accounts, see [Segre, 1994, pp. 307–313]. I am grateful to an anonymous referee for pointing out [Sieg, 2014] regarding the comparison between Peano's axiomatisation and Hilbert's early work on geometry.

[27]On Dedekind's structuralism, see [Reck, 2003], [Yap, 2009], [Sieg; Schlimm, 2014], [Sieg; Morris, 2018] and [Ferreirós; Reck, 2020].

> which should be noticed immediately. Here the number is not defined, but its fundamental properties are stated. Dedekind defines the number instead, and specifically calls number what satisfies the aforementioned conditions. Evidently the two things coincide. [Peano, 1891d, p. 88]

Although Peano's requirements regarding the symbolisation of mathematical theories should not be underestimated – it shall be shown below to have a great importance – soon after he had achieved an axiomatisation of arithmetic, he assumed that the axioms played a similar role to the conditions of a general definition such as Dedekind's.[28]

In 'Sul concetto di numero' [1891d], a clear example of the similarity between the role played by Peano's axioms and Dedekind's definition can be found. Following Dedekind's account in *Was sind und was sollen die Zahlen?* [1888] – a work that he quotes several times in this article – Peano offers a structuralist interpretation of the axioms:

> These propositions express the necessary and sufficient conditions so that the entities of a system can be put in univocal correspondence with the series of N; and can be stated as follows:
> (1) A specific entity of the system is given the name 1.
> (2) Let an operation be defined such that to every entity a of the system there is a corresponding one, $a+$, also of the system.
> (3) And that two entities, whose corresponding [entities] are equal, are equal.
> (4) The entity named 1 is not the corresponding [entity] of any.
> (5) And finally that it [N] is the class common to all the classes s that contain the individual 1, and that, when they contain an individual, they also contain its corresponding [entity]. [Peano, 1891d, p. 87]

In this informal rendering of the primitive propositions of arithmetic, Peano characterises a system whose structural properties are those of the ordering of natural numbers. In fact, he suggests that there might be systems, other than the set of natural numbers which satisfy the axioms. However, it should be observed that the system of natural numbers, N, is assumed to be given. In other words, this system is not created by abstraction from the systems that satisfy the structural properties expressed by the primitive propositions.

In later works Peano displays a more explicit structuralist position. In the second part of the second volume of the *Formulaire de mathématiques*, he also defends the idea that the the axioms of arithmetic set the conditions necessary for identifying a class of structured sets: any collection of objects ordered in such a way that satisfies the axioms has the structural properties of the set of natural numbers. Peano states the following:

> These Pp [Primitive propositions], the necessity of which we have seen, are sufficient to deduce all the properties of the numbers which we will meet in the following. But there is an infinite number of systems which satisfy all the Pp. For example, they are all verified

[28] On a comparison between Dedekind's and Peano's work on the foundations of arithmetic, see [Borga *et al*, 1985, pp. 105–116].

> if we replace N_0 and 0 with N_1 and 1 [...].[29] All the systems which satisfy the 5 Pp are in mutual correspondence with the numbers. The number, N_0, is what is obtained by abstraction from all these systems; in other words, the number is the system which has all and only the properties stated by the 5 primitive P [propositions]. [Peano, 1898, p. 2]

In this context, abstraction is not understood by Peano as a process that creates or reifies some abstract entities – in this case, the natural numbers. Peano sees abstraction as more than a creative process; rather, it is a limitation of the properties of the elements of the systems considered. Therefore, by appealing to abstraction, Peano implies that one of the systems which satisfies the axioms stands out as one which only has the properties stated in the primitive propositions. In this sense, N_0 is a particularly adequate representative of the class of systems that satisfy the axioms.[30]

Textual support can therefore be found for the claim that Peano fulfils the second requirement of a minimal methodological structuralism (see page 2). Furthermore, Peano states that the systems that satisfy the axioms "can be put in univocal correspondence with the series of N" and "are in mutual correspondence with the numbers" [1891d, p. 87]. He also acknowledges that the axioms "are sufficient to deduce all the properties of the numbers" [1898, p. 2]. These claims could be seen as implicit acknowledgements of the categoricity and semantic completeness, respectively, of the axiom system of the natural numbers. Peano does not justify any of these remarks, but at the very least they show awareness of metatheoretical considerations that became relevant in foundational studies of late nineteenth-century and early twentieth-century mathematics.

Moreover, the idea that there is an infinite number of structured systems which satisfy the axioms of arithmetic, and that the system of natural numbers is determined by abstraction from all these systems, strengthens the claim that Peano also satisfies the first requirement of methodological structuralism.

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[29] Note that 'N_0' refers to the set of natural numbers and 'N_1' to the set of positive integers.

[30] On Peano's notion of abstraction, see [Borga *et al*, 1985, pp. 127–129], [Segre, 1994], [Mancosu, 2018] and [Cantù, 2021].

Despite the similarities between Peano's account of the systems that satisfy the axioms of arithmetic and Dedekind's remarks on the class of simply infinite systems, I do not want to support or reject the claim that Dedekind influenced Peano's notion of abstraction. In this section, I provide textual evidence in support of the claim that Peano explicitly considers a multitude of systems that exhibit the relational features stated in the axioms of arithmetic. Peano appeals to the notion of abstraction as a means to characterise the relationship between the system of natural numbers and any other system that satisfies the axioms. However, this does not imply that his notion of abstraction and Dedekind's are analogous. In fact, there is no consensus in the secondary literature on Dedekind's notion of abstraction. While some studies defend an ontologically substantive sense of abstraction, on account of which, e.g., the system of natural numbers is created (see [Reck, 2003], [Yap, 2009]), others argue for a less ontologically involved sense (see [Sieg; Schlimm, 2005], [Sieg; Schlimm, 2014], [Sieg; Morris, 2018]). I am indebted to an anonymous referee for raising the issue of the relationship between Peano's and Dedekind's notions of abstraction.

4.2. **Higher-level definition vs schematic axiomatisation.** Peano's structural understanding of his axiomatisation of arithmetic can be connected to Dedekind's definition of a simply infinite system from two different perspectives. On the one hand, Peano's account of the role of the axioms of arithmetic seems to put him in almost complete allegiance with Dedekind's position in *Was sind und was sollen die Zahlen?* [1888]. In the aforementioned passages, we can find explicit recognitions by Peano that there is a multitude of structured systems which share the relational features expressed by the axioms of arithmetic. On the other hand, Peano rejects the definition of the primitive notions of arithmetic, whilst Dedekind offers explicit definitions for them.

As a means to clarify the disagreement concerning the explicit definition of the primitive notions of arithmetic, and after having considered Peano's axiomatisation, it is convenient to pay attention to Dedekind's definition of a simply infinite system. The latter relies on the basic notions of his theory of classes:

> Definition. A system N is said to be *simply infinite* when there exists a similar mapping ϕ of N into itself such that N appears as the chain (44) of an element not contained in $\phi(N)$ [...]. [T]he essence of a simply infinite system N consists in the existence of a mapping ϕ of N and an element 1 which satisfy the following conditions α, β, γ, δ:
>
> α. $N' \ni N$.
>
> β. $N = 1_0$.
>
> γ. The element 1 is not contained in N'.
>
> δ. The mapping ϕ is similar.
>
> [Dedekind, 1888, Sect. 71, p. 359]/[Ewald, 1996, p. 808]

In this definition, the symbols 'N', 'ϕ' and '1' are variables which are implicitly bound. Accordingly, they do not refer to a specific system, mapping, or element respectively, but express generality over such entities. The primitive notions of Dedekind's theory of classes are exactly these entities, which are not defined but elucidated in *Was sind und was sollen die Zahlen?* [1888, pp. 344-348]/[Ewald, 1996, pp. 796–800]. The class-theoretical notions are thus substantially linked to the variables 'N', 'ϕ' and '1' by means of these elucidations. In this sense, Dedekind's definition conveys a specific content in terms of systems, mappings, and elements of these systems, and therefore cannot be considered a formal or schematic definition.[31]

In contrast, the primitive terms in Peano's axiomatisation of arithmetic are not included in the axioms as bound variables (see Section 3.2); they should rather be seen as non-logical constants associated with a standard interpretation. As a result, the systems that satisfy the axioms are not obtained by means of an instantiation from a general definition as in Dedekind's case, but as the result of considering alternative interpretations of the axioms.

[31]This interpretation of Dedekind's definition of a simply infinite system agrees with Klev's account [2011, pp. 647–655, 665–671]. See also [Sieg; Schlimm, 2014, pp. 300–301, 311]. On the diverging interpretations of Dedekind's definition in the secondary literature, see Footnote 30.

It could be said that Dedekind provides a *definition of a higher-level notion*, while Peano offers a *schematic axiomatisation*.[32]

The claim that Peano offers a schematic axiomatisation of arithmetic does not imply that the axioms should be understood as schemata, but rather relies on the use of non-logical constants which are not subject to universal quantification. This idea can be better understood in the context of Peano's independence arguments. In the third part of the second volume of the *Formulaire de mathématiques*, immediately after presenting the axioms, Peano proposes an evaluation of their independence:

> In order to recognise the independence of a Pp [primitive proposition], it is enough to give an interpretation to the primitive symbols 0, N_0, +, such that all Pp which are different from the one considered are verified.
>
> The absolute independence of Pp 1–5 is proved by the following 5 examples:
>
> (1) If, by attributing to 0 and + the usual meaning, N_0 is given the meaning 'positive integer', indicated in what follows by N_1, all conditions ·2–·5 but ·1 are verified.
> (2) If by preserving the usual meaning of 0 and +, N_0 is given the meaning 'digit' or the set of numbers $0, 1, 2, \ldots, 9$, all conditions except ·2 are satisfied.
> (3) All conditions, save ·3, will be satisfied by a periodic system, preceded by an anti-period; such as the sequence $0, 1, 1, 1, \ldots$
> (4) A periodic system, e.g. the astronomic hours of the day, where the hour that follows 23 is 0, does not satisfy [condition] ·4.
> (5) Let 0 and N_0 have the usual meaning, and let $a+$ have the value $a+2$. Conditions ·1–·4 are satisfied, but not ·5, because if s is replaced with 'even number, $2\mathrm{N}_0$', the Ths [thesis, i.e., consequent of ·5: $\mathrm{N}_0 \supset s$] is not true. [Peano, 1899, p. 30]

The primitive terms of arithmetic used in the symbolisation of the axioms, namely, '0', 'N_0', and '+', are to be understood as non-logical constants in these examples. This is a natural move if these symbols – which are associated with a standard interpretation – do not occur in the axioms as bound variables. In the derivation of theorems and, crucially, in the context of independence arguments, the original meaning of these symbols is irrelevant (see sections 3.2 and 5.1). Despite Peano's insistence about the expression in the primitive propositions of the properties of the natural numbers, once the axioms are selected and formulated, the proof of arithmetical theorems and the evaluation of the deductive relations between the axioms do not require the primitive terms to be interpreted according to their original meaning. In

[32] I take this terminology from [Klev, 2011, p. 651]. I agree with Klev [2011, pp. 650–653] regarding the convenience of distinguishing Dedekind's point of view, based on the construction of arithmetic starting from definitions, and Peano's (or Hilbert's in Klev's discussion), which has axioms as the starting point. In contrast, Ferreirós claims that there is no essential difference between Peano's and Dedekind's points of view as long as there is a common logical background consisting in basic set theory [2009, p. 49]. See also [Ferreirós; Lassalle-Casanave, 2022]. Sieg's [2014] characterisation of Hilbert's existential structuralism is, in certain relevant respects, analogous to Peano's schematic axiomatisation, and yet Sieg – with Ferreirós – claims that "Hilbert's [1899] axioms characterize fully continuous systems in analogy to the way in which Dedekind's conditions characterize simply infinite ones in Dedekind (1888)" [2014, p. 136]. See also [Sieg; Schlimm, 2014].

particular, in the context of independence arguments the primitive terms can be reinterpreted.

The reinterpretation of mathematical terms was fairly common in nineteenth-century mathematics. It often consisted in a generalisation of the original meaning of the reinterpreted terms, or in a restriction of their generality. For example, in the Preface to the first edition of *Was sind und was sollen die Zahlen?* [1888], Dedekind suggests a reinterpretation of the notion of point according to which the ratios of the distances of any two points are algebraic numbers [1888, pp. 339–340]/[Ewald, 1996, p. 793]. The purpose of this reinterpretation is to show that the notion of space does not require continuity in all contexts: the space determined by the points in Dedekind's reinterpretation is discontinuous and yet "all constructions that occur in Euclid's *Elements* can [...] be just as accurately effected here as in perfectly continuous space" [1888, pp. 339–340]/[Ewald, 1996, p. 793].

The development of projective geometry in the second half of the nineteenth century relied, at least partly, on a reinterpretation of the terms 'point' and 'straight line' by means of duality principles. Projective geometry was one of the main lines of development for the Italian school of algebraic geometry and also for Peano and his circle.[33] Another prominent example of reinterpretation in nineteenth-century mathematics is the enlargement of the number domain with new kinds of numbers. This enlargement required the redefinition of arithmetical operations and the reinterpretation of their symbols depending on the kind of numbers involved. Each numerical domain demands a specific definition of the mathematical operations and relations involved, and the symbols referring to these operations and relations are reinterpreted according to their application. Peano was well aware of this strategy and reinterpreted some of the symbols of his mathematical logic to adapt them to the different contexts of their application. For instance, in the first volume of the *Formulaire de Mathématiques*, Peano provides definitions of identity between propositions [1895, §1, p. 1], between classes [1895, §4, p. 5], and between functions [1895, §5, p. 6]. In all these cases, Peano relies on the generality of the notion of identity to define specific cases.

As we shall see in Section 5.3, what distinguishes Peano's interpretation of the primitive terms in the context of independence arguments is the fact that the alternative interpretations suggested are not necessarily generalisations or particular cases of the standard. They are motivated by an abstract understanding of the axioms, according to which the primitive terms are not attached to any particular meaning and in this sense any reinterpretation can involve a switch in the ontology established by the relevant mathematical theory. Moreover, the fact that the axioms are fully symbolised allows Peano to precisely characterise a reinterpretation of those axioms as the result of a reinterpretation of the primitive terms, while the meaning of the logical and class theoretical symbols remains intact. I shall explain in Section 5.3 that

[33] On Peano's importance in the debate concerning the relationship between planar and solid geometry, specially in a projective context, see [Arana; Mancosu, 2012]. On the development of nineteenth-century geometry and its relation to independence proofs and the pre-history of model theory, see [Blanchette, 2017, pp. 47–48], [Eder, 2019], [Eder; Schiemer, 2018], Eder [2021], [Tappenden, 1997] and [Webb, 1995].

Peano takes full advantage of this strategy regarding the reinterpretation of the primitive terms in his independence proofs of the axioms of geometry.

There is yet another element that sets Peano's axiomatisation of arithmetic apart from Dedekind's definition of a simply infinite system. In his axiomatisations of arithmetic of [1889a], [1891d] or in those of the successive editions of the *Formulaire de Mathématiques*, Peano does not investigate the nature of the class of structured sets that satisfy the axioms, and which are considered as possible interpretations in the independence arguments. As has been noted in the previous section, Peano shows awareness of categoricity and the semantical completeness of his axioms system, but he does not pursue a systematic investigation of these metatheoretical questions. In contrast, the justification of the fact that all simply infinite systems are isomorphic is Dedekind's purpose in Sections 132–134 of *Was sind und was sollen die Zahlen?* [1888, pp. 376–378]/[Ewald, 1996, pp. 821–823].[34] Once the primitive propositions are established, Peano seems to view them as formal axioms from which theorems are obtained by means of logical inference. The specific meaning of the axioms and, in particular, the structural properties they are claimed to define, are left aside once Peano starts drawing inferences from the axioms and the definitions of the derived notions of arithmetic. In this context, Peano's methodological structuralism is limited, on the one hand, to the recognition that the axioms of arithmetic can be satisfied by (an infinite number of) systems of objects and that these systems are, in some intuitive sense, isomorphic; and, on the other hand, an implicit acknowledgement of the semantic completeness of these axioms. The modest scope of Peano's structuralist position could be explained, both by the fact that, for him, the primitive notions of arithmetic are specific entities and are accessible to any mathematician; and also by his idea that the symbolisation of arithmetic is a goal by itself, and not a means to further develop this theory.[35]

Peano's attitude towards metatheoretical investigations contrasts not only with Dedekind's perspective, but also with Hilbert's, for whom metatheory was the main focus of interest. For Hilbert, the proof of the independence and consistency of the axioms or the completeness of a mathematical theory

[34]In this regard, as one of the referees noted, the scope of Dedekind's metatheoretical investigations is also limited. Dedekind proves the categoricity of his definition of a simply infinite system, but – similarly to Peano – he suggests the completeness of this definition without justification [1888, pp. 377-378]/[Ewald, 1996, p. 823]. See [Awodey; Reck, 2002, pp. 6–8] and [Sieg; Schlimm, 2005, pp. 152–154].

[35]The idea that Peano's metatheoretical work is – besides passing remarks on what could be understood as categoricity and completeness – limited to independence proofs has also been considered in [Avellone *et al*, 2002]. The proof of the independence of the axioms and the recognition of the existence of alternative systems that satisfy the axioms could be seen by Peano as the final step of his symbolisation, and not as the starting point of metatheoretical investigations. As Pieri – who studied both with Peano and with Segre – states, the independence of the axioms is seen as a requirement of the logically perfect presentation of a theory [1906, p. 201]. See also [Borga *et al*, 1985, pp. 64–65]. That said, Peano reflects on the consistency of the axioms of arithmetic in [1906]. See [Lolli, 2011, pp. 60–63].

are the fundamental questions of mathematical research, and he put the solution of these questions at the forefront of his work.[36]

5. The schematic understanding of the axioms and the birth of formal languages

Peano's work on the symbolisation of mathematical theories, with rigour and precision in the expression of mathematical laws as its cornerstone, obeys methodological principles such as the use of the minimal possible set of primitive notions and axioms. Peano believes that logically perfect presentations of mathematical theories also involve the irreducibility of the primitive notions of these theories and the independence of the axioms.

As we have seen in the previous section, Peano's understanding of the axioms involves the recognition of a multitude of systems which satisfy them, and some of these systems are explicitly considered in his independence arguments. The justification of the independence of the axioms in a specific symbolisation of a mathematical theory – namely, geometry and arithmetic – requires Peano to first disregard the meaning of the primitive terms which occur in the axioms and secondly, to develop a collection of technical tools related to the assumption of a model-theoretic point of view.

In this section, the importance of the technical tools developed by Peano in his proofs of independence shall be considered. First, Peano's view on the use of axioms in the derivation of theorems shall be put in historical context. Second, a collection of notions – namely formalisation, formal language, and the model-theoretic point of view – required for a systematic evaluation of Peano's independence arguments shall be characterised. Third, the connection between Peano's schematic understanding of the axioms and the methodology he devised to demonstrate their independence shall be evaluated. Finally, I shall discuss whether, in his independence proofs, Peano lays down the basic components of a formal language.

5.1. **Peano's deductivism.** The investigation of the deductive relations between the axioms brings into focus an important aspect of Peano's view on the nature of symbolised mathematical theories. For him, the derivation of the theorems of a mathematical theory, once the axioms and definitions of derived notions are formulated, is a purely logical procedure with no

[36]See, for instance, Hilbert's account of the motivation of his early work on geometry in a letter to Frege, dated December 29, of 1899:

> It was of necessity that I had to set up my axiomatic system: I wanted to make it possible to understand those geometrical propositions that I regard as the most important results of geometrical inquiries: that the parallel axiom is not a consequence of the other axioms, and similarly Archimedes' axiom, etc. I wanted to answer the question whether it is possible to prove the proposition that in two identical rectangles with an identical base line the sides must also be identical, or whether as in Euclid this proposition is a new postulate. I wanted to make it possible to understand and answer such questions as why the sum of the angles in a triangle is equal to two right angles and how this fact is connected with the parallel axiom. [Frege, 1976, p. 65]/[1980, pp. 38–39]

See also [Hilbert, 1900]. On the metatheoretical orientation of Hilbert's work, see [Hallett, 1994], [Sieg, 2014] and [Eder; Schiemer, 2018].

reference to the specific meaning of the propositions involved. In 'Formules de logique mathématique', he states that "a demonstration is reduced to a series of transformations, according to the rules mentioned [logical rules of reasoning] [...]. These transformations are analogous to algebraic rules for solving a system of equations" [Peano, 1900, p. 322].[37]

Peano echoes an approach that was prominent in late nineteenth-century mathematics: deductivism. It had a significant role in the development of non-Euclidean geometry. Advocates of deductivism defend the elimination of all traces of intuition in mathematical proofs by disregarding the meaning of the symbols in deductions. Pasch expresses his deductivist position in projective geometry in *Vorlesungen über neuere Geometrie*:

> In fact, if geometry is to be genuinely deductive, the process of deducing must everywhere be independent of the *sense* of geometrical concepts, just as it must be independent of figures; only the *relationships* between the geometrical concepts put down in the sentences used – respectively, definitions – should come into consideration. [Pasch, 1882, p. 98]

The relationships between concepts are taken to be logical and thus are established by means of logical laws.[38] Pasch does not make explicit the logical laws involved in mathematical deductions but this is precisely one of the main aspects involved in Peano's symbolisation of mathematical theories.[39]

5.2. **Formalisation and the model-theoretic point of view.** Before we consider in detail the nature of Peano's independence arguments, it shall be useful to clarify some terminology that shall become instrumental in the discussion that follows. The notions of formalisation, formal language, and the model-theoretic point of view shall be characterised, and their mutual relations clarified.

An integral part of a formalisation is the use of a formal language. A *formal language* is composed of a set of logical symbols, a set of non-logical symbols (including variables and non-logical constants) and (possibly) a set of auxiliary symbols. What characterises a formal language is, first, a syntax that allows the definition of notions such as well-formed formula or atomic formula by means of formation rules; second, a set of semantic conventions that establish how well-formed formulas denote and how to evaluate them. These elements presuppose the distinction between object language (i.e., the formal language) and metalanguage (the language in which semantic rules

[37]See similar claims concerning the derivation of arithmetical theorems [Peano, 1889a, p. 21]/[1973, p. 102] and geometrical theorems [Peano, 1889b, p. 81].

[38]See also [Pasch, 1882, p. 17]. On Pasch's deductivism, see [Gandon, 2005] and [Schlimm, 2010, pp. 102–107]. As Schlimm [2010, p. 103] states, soon after the publication of [Pasch, 1882], the deductivist position defended there was considered common by Klein.

[39]Although Peano provides presentations of his mathematical logic from 1888 to the last editions of the *Formulaire de mathématiques* and tries to make explicit the logical principles involved in mathematical proofs and definitions, he never characterises the notion of deduction. That said, in his mature works on logic in the late 1890s, he progressively advances towards a fully deductive approach. I considered this matter in more detail in [Bertran-San Millán, 2021].

are expressed).[40] I take the language of first-order logic as the paradigmatic example of a formal language. By '*formalisation*' I mean the translation of a set of sentences expressed in a language L into a corresponding set of sentences expressed in a formal language L'. Sometimes, the language L is just a natural language (possibly extended with the non-logical constants of a scientific language), but it is often a language with some degree of symbolisation such as a mathematical language. The translation from L into L' preserves the syntactic status of the terms of L, and reformulates the sentences of L using the logical symbols and the non-logical constants of L'.[41]

The emergence of formal languages and the formalisation of mathematical theories is usually associated with a specific perspective, which can be called the '*model-theoretic point of view*'.[42] There are three different elements that characterise this point of view. First, it requires a formal language, and specifically, a set of uninterpreted non-logical constants. Second, it assumes the notion of interpretation, which involves a universe of discourse over which the variables range, and the possibility of assigning different meanings to the non-logical constants. Third, it involves the notion of the satisfaction of a formula in a specific interpretation, which in turn is dependent on the semantic conventions of the formal language.

5.3. **Independence arguments and the schematic understanding of the axioms.** From Peano's point of view, the symbolisation of mathematical theories involves, as a complementary step to their axiomatisation, the justification of the independence of the axioms. This step moves his approach to a metamathematical level. Peano adopts a deductivist stance and understands the axioms as abstract postulates which can be assigned different interpretations; this is a move which in turn is instrumental to the solution of metatheoretical questions such as the independence of the axioms.

The best witnesses to Peano's strategy for proving the independence of the axioms can be found in his works on geometry. In fact, Peano first considers the independence of the axioms of a mathematical theory in *Principii di Geometria*. As we saw in Section 3.2, in the discussion that follows his axiomatic presentation, Peano introduces an abstract understanding of the primitive terms [1889b, p. 77]. By introducing such an abstract understanding, Peano is making it possible to consider interpretations of the primitive terms that involve a change in the ontology established by elementary geometry. As a means to acquire a meaning that is independent of their original interpretation, the primitive terms are severed from their

[40] On the notion of formal language, see [Church, 1956, pp. 2–68]. According to the properties described above, I assume here that a formal language is an inductively generated set of expressions determined by a vocabulary. For an alternative view of the expressions of a formal language, see [Sundholm, 2002].

[41] Again, I do not claim to characterise a formalisation or a formal language in a standard way. My main purpose is to fix a specific sense for these terms that can be recognised by the reader, and to establish a secure ground for the historical analysis that follows.

[42] I take the phrase 'model-theoretic point of view' from [Demopoulos, 1994]. I essentially share Demopoulos' framework, even if I articulate it in a different way. See, in particular, [Demopoulos, 1994, p. 213]. [Eder, 2019, pp. 5549–5550] and [Badesa, 2004, pp. 59–60] offer similar accounts.

standard geometrical meaning and seen as abstract symbols, i.e., as symbols with no specific meaning. After the passage quoted in Section 3.2, page 10, the following can be found:

> Depending on the meaning attributed to the undefined signs **1** and $c \epsilon ab$, the axioms may or may not be satisfied. If a certain group of axioms is verified, all the propositions that are deduced [from them] will also be true, since these propositions are but transformations of those axioms and definitions. [Peano, 1889b, p. 77]

Peano states that the meaning attributed to the primitive terms determines the semantic value of the axioms, which in turn determines – assuming what in contemporary terms would be referred to as the soundness of the calculus – the semantic value of the theorems.

In *Principii di Geometria*, Peano does not systematically evaluate the independence of the axioms he has introduced, although he is certainly in possession of all the tools required for that task. He provides instead some semantic considerations that put the focus on the satisfaction of some or all of the axioms of linear geometry by means of interpretations that set a domain for the variables and a meaning of the primitive terms.[43] One of the remarkable elements in Peano's account is the fact that he considers interpretations of the axioms of planar geometry that do not belong to geometry, but to analysis. See, for instance, Peano's remarks about the satisfaction of Axiom V:

(V) $$a, b \epsilon \mathbf{1} \,.\, \supset .\, ab = ba,$$

> Axiom V says that the segment ab is a symmetric function of a and b. Not every relation put in place of $c \epsilon ab$ is symmetric with respect to a and b. If a, b, c are numbers, and $f(a, b)$ is a symmetric function of a and b, the relation $(a-b)^2c = f(a, b)$ satisfies all the axioms stated so far. [Peano, 1889b, p. 84]

Peano interprets Axiom V as follows:[44]

$$a, b \epsilon \mathbf{1} \,.\, \supset .\, c \epsilon ab =_c c \epsilon ba.$$

[43] Peano states that according to one of these interpretations, "all the axioms preceding X stand[.]

(X) $$a, b \epsilon \mathbf{1} \,.\, c, d \epsilon a'b : \supset : c = d \,.\cup.\, d \epsilon bc \,.\cup.\, c \epsilon bd.$$

[T]his one [axiom X], depending on the case, may be true, or not; so it is not a consequence of the precedent" [Peano, 1889b, p. 88]. In connection with this, Mancosu [Mancosu *et al*, 2009, p. 329] claims that the first application of the method for providing proofs of independence can be found in *Principii di Geometria*.

In contrast, Church [1956, p. 328, fn. 539] singles out [Peano, 1891d] as one example of the proof of the independence of a postulate by providing an interpretation that satisfies the remaining postulates and does not satisfy a formalisation of the postulate in question. Church also acknowledges that the origin of Peano's method for the proof of the independence of postulates can be traced to Bolyai and Lobachevsky's work on the independence of Euclid's parallels postulate.

[44] Note that, according to this interpretation, '=' should be read as a biconditional. This is justified by the following logical law, which Peano includes in *Arithmetices principia* [1889a, p. xi]/[1973, p. 108]:

(51) $$a, b \epsilon \mathrm{K} \,.\, \supset \therefore a = b : = : x \epsilon a \,.\, =_x .\, x \epsilon b,$$

where '$a \epsilon \mathrm{K}$' means that a is a class.

Then, the reinterpretation of the primitive terms of (V) consists of establishing a new domain for the letters and a new interpretation of the term '$c \epsilon ab$'. According to Peano's example, to '$c \epsilon ab$' – which, in this context, refers to a ternary relation that is applied to c, a, and b (as if the class membership relation ϵ and the segment formation function formed a single relation) – a class of interpretations is assigned, determined by $(a-b)^2c = f(a,b)$. In other words, the circumstance that a point c belongs to the segment determined by a and b is replaced with the equation $(a-b)^2c = f(a,b)$, where f is any function such that $f(a,b) = f(b,a)$, for any a, b, as the interpretation of '$c \epsilon ab$'. The domain of the letters 'a, 'b', 'c' is the set of real numbers, and any non-logical relation or function symbol is interpreted as a relation or operation between numbers. This involves, in particular, the reinterpretation of the membership relation symbol 'ϵ', which belongs to the language of the calculus of classes; its original meaning is dissolved in $(a-b)^2c = f(a,b)$. That is, even if we consider a particular function in the place of f, no component of $(a-b)^2c = f(a,b)$ can be univocally seen as the interpretation of 'ϵ' in '$c \epsilon ab$'.[45]

The kind of interpretation suggested in this example involves a level of sophistication uncommon in the consideration of the semantic value of the axioms of geometry in late nineteenth-century mathematics. As suggested in Section 4.2, a reinterpretation usually involves either a generalisation of the original meaning of mathematical terms or a restriction on their generality. The interpretation of points as arbitrary numbers is, *prima facie*, neither.[46] This example, in fact, involves a switch in the ontology determined by elementary geometry, which takes place after the primitive terms are severed from their original meaning. It could be said that the process of reinterpretation suggested in Peano's example amount to, as an intermediate step, the fact that the primitive terms cease to be substantive. If the process of selection of the primitive terms and the formulation of the geometrical axioms, on the one hand, and the derivation of theorems and the justification of the independence of the axioms, on the other, are kept separate, and if we focus only on the latter, then the reinterpretation of the primitive terms could be seen as the assignation of a interpretation – which can be completely unrelated to the standard interpretation established in the former phase – of uninterpreted non-logical constants.

Peano develops his approach to the proofs of independence of the axioms of geometry in 'Sui fondamenti della Geometria' [1894a]. In this work, he applies explicitly and systematically what can be called the 'method of

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[45]Note that the membership relation symbol also occurs in the antecedent of Axiom V and yet that occurrence is not reinterpreted.

[46]Examples of interpretation, such as the aforequoted, set a precedent for Hilbert's *Grundlagen der Geometrie* [1899]. See [Kennedy, 1972]. It is worth noting that Peano proposes interpretations in real analysis of *symbolised* axioms of geometry, while Hilbert's axiomatisation is expressed in natural language.

Although Hilbert's work on metatheoretical questions, such as the independence of the axioms of geometry, could have benefitted from the work in this field done by Peano's school, he seemed to be unaware of the latter's value. On Hilbert's attitude towards Peano's school, specifically Padoa, see [Cassari, 2011, pp. 152–153].

exemplification'.[47] Once the axioms have been symbolised, a new interpretation of the primitive symbols of the theory is first provided, and then it is tested whether such an interpretation satisfies all the axioms but one. Peano characterises the method of exemplification in the following way:

> One can prove the independence of some postulates from others by means of examples. The examples intended to prove the independence of the postulates are obtained by attributing any meaning at all to the undefined signs, which here are the point, and the relationship between three points expressed by $c \,\varepsilon\, ab$; and if it is found that the fundamental signs, according to this new meaning, satisfy a group of primitive propositions, and not all of them, it will be deduced that the latter are not necessary consequences of the former [...].
>
> So to prove the independence of n postulates, we should provide n examples of interpretation of undefined signs (in our case p, and $c \,\varepsilon\, ab$), each of which satisfying $n-1$ postulates, and not the rest. [Peano, 1894a, p. 127]

To evaluate the independence of the first eleven postulates (the axioms of linear geometry), Peano proposes seven interpretations [1894a, pp. 128–129]. Some of them do not belong to geometry; Peano considers interpretations that involve rational numbers, real numbers, and integers among others. Nevertheless, the list of examples does not suffice to prove the independence of all eleven postulates; Peano acknowledges that the proof of independence for some of them is lacking [1894a, pp. 129].

As suggested in Section 3.2, Peano's procedure for the proof of the independence of the axioms of a mathematical theory consists in two phases. First, he provides a symbolised presentation of the theory which, using the primitive terms, includes its axiomatisation and the definition of derived notions. In that symbolisation, Peano keeps the logical and mathematical symbols separate. Second, the primitive terms are understood as abstract symbols that can acquire multiple interpretations. Each interpretation determines a domain for the variables and a meaning of the primitive terms and, as a result, the axioms – seen as postulates – can acquire a semantic value. It is then established whether each axiom is satisfied by each interpretation. If this second phase is isolated from the first, Peano's remarks, that no meaning is attached to the primitive symbols, can be understood as a tacit assumption that the primitive terms are uninterpreted non-logical constants that acquire several interpretations in the independence arguments. All in all, Peano's strategy involves the deployment of crucial techniques required for the formalisation of mathematical theories and, to a great extent, the acquisition of a model-theoretic point of view.

[47] It was not uncommon in the late nineteenth and early twentieth centuries to refer – as Peano does – to the interpretations of the axioms that are used in independence arguments as 'examples'. In the first volume of *Vorlesungen über die Algebra der Logik*, Schröder states that the impossibility to demonstrate a proposition from others – i.e., to prove its independence – can be shown by means of exemplification (*Exemplifikation*) [1890, pp. 286–287]. Similarly, Hilbert and Huntington refer to the interpretations that satisfy a group of axioms of geometry as 'examples'. See [Hallett; Majer, 2004, p. 306] and [Huntington, 1913, pp. 548–554], respectively. See also [Peano, 1899, p. 30].

The method of exemplification adopted by Peano, or proofs of independence of the axioms of geometry in general, are not really uncommon in late nineteenth-century geometry. Peano's novelty is the justification of the independence of *symbolised* axioms, and the exploitation of this symbolisation to establish a clear connection between the interpretation of the primitive terms and the satisfaction of the axioms. In this sense, Peano's development of a logical language and a logical apparatus that can be effectively used for the resolution of metatheoretical questions such as the independence of the axioms of arithmetic and geometry is unprecedented in the late 1880s. Peano's metatheoretical contributions would be impossible without the symbolisation of geometry and arithmetic, the separation between logical and non-logical symbols that these symbolisations involve, and the use of non-logical constants. After Peano's work, model theory proper lays just a step ahead; a step which consists in the explicit formulation of a fully formal language.

5.4. **The birth of formal languages.** In the previous section, I analysed Peano's use of some of the primitive symbols of the language of the calculus of classes as non-logical constants. Peano had established a clear distinction between logical and mathematical distinctions since his first expositions of mathematical logic. The abstract understanding of a primitive term such as '$c \epsilon ab$' – which includes a class-theoretical symbol – can be seen as a further step towards the formalisation of the calculus of classes and the creation of a formal language. As a means to determine whether Peano was ready to take that step, his account of the independence of the axioms of the calculus of classes shall be studied. Peano's methodology in these proofs can be connected with the algebra of logic tradition; specifically, it can be linked with Schröder's axiomatisation of the calculus of classes and compared to Löwenheim's construction of a formal language in the second decade of the twentieth century.

In a lecture given at the 53rd Meeting of the British Association for the Advancement of Science [1884a], Schröder announced that one of the distributive laws was independent of the calculus of classes, claiming that this law could not be proved from the definitions of the sum, the product, the modules 0 and 1, and the properties of the subsumption. Although by 1884 the calculus of classes had not yet been axiomatised, Schröder, in a parallel lecture at the same congress, offered an overview of a "calculus with algorithms or calculesses" [1884b]. In the first volume of *Vorlesungen über die Algebra der Logik* [1890], he axiomatises the calculus of classes and shows the independence of the aforementioned distributive law from the first seven axioms of the calculus – in fact, showing the independence of two distributive laws – by providing two interpretations that satisfy the axioms and do not satisfy the laws [1890, Anhänge 4–6, pp. 617–699].[48]

Schröder's proof is the first application of an independence proof to the calculus of classes. The importance of this fact lies in the field in which the method of independence proofs is applied: in the late nineteenth century the

[48]On Schröder's calculus of classes and the independence of the distributive laws, see [Huntington, 1904, p. 291, 297–305]. On Schröder's proof of independence, see [Badesa, 2004, pp. 21–25], [Mancosu *et al*, 2009, p. 375], [Peckhaus, 1994] and [Thiel, 1994].

calculus of classes was taken to be part of logic, as Peano witnessed in his presentations on mathematical logic.[49]

Peano knew Schröder's proof; he praised it in his review of the first volume of *Vorlesungen über die Algebra der Logik* [Peano, 1891e, p. 116]. Actually, in a footnote, Peano reconstructs Schröder's argument of the independence of the law $(a \cup b)c \supset ac \cup bc$ from the other axioms of Schröder's calculus of classes. In this reconstruction, Peano uses the resources of his mathematical logic. In a preliminary step, Peano introduces the primitive terms of the calculus of classes as uninterpreted symbols:[50]

> Consider a system of entities S; suppose that a relation between entities a and b, which we will denote by $a \supset^* b$, is defined (the sign $\supset^*$ refers to any relation; we keep the sign $\supset$ to denote deduction). Suppose that it is *reflexive* and *transitive* [...], i.e., that we have
>
> $$a \supset^* a$$
>
> $$a \supset^* b \,.\, b \supset^* c : \supset . \, a \supset^* c.$$
>
> Then suppose that given two entities a and b of the system, an entity $a \cap^* b$ is determined such that, whatever c is, we have:
>
> $$c \supset^* a \cap^* b \,. = : c \supset^* a \,.\, c \supset^* b$$
>
> and another entity $a \cup^* b$ such that
>
> $$a \cup^* b \supset^* c : = : a \supset^* c \,.\, b \supset^* c.$$
>
> [Peano, 1891e, p. 117]

There is a clear effort by Peano towards the formalisation of the propositions which express the basic properties of class inclusion, union, and intersection. Note that there is no indication of the domain involved; Peano does not fix a specific ontology, and possible interpretations of the letters and of the terms considered are left undetermined. In the context of the proof of the independence of the distributive law, the class theoretical terms '$\supset$', '$\cup$' and '$\cap$' are replaced with the non-logical constants '$\supset^*$', '$\cup^*$' and '$\cap^*$', respectively. The use of specific notation – or, better, the use of the asterisk as a distinctive notational device – helps Peano distinguish interpreted and uninterpreted symbols. In fact, directly after the passage quoted above, Peano provides possible interpretations of '$\supset$', '$\cup$' and '$\cap$' which satisfy the properties stated in the passage and the distributive law. As a result, the formulas included in the last quote should be seen as a formalisation, or at the very least, as a proto-formalisation.

[49] Peano never offered a characterisation of the notion of logic. That said, from his very first presentation of a system of logic in *Arithmetices principia* to his mature publications on the subject, he always divides mathematical logic into the calculus of classes and the calculus of propositions. Moreover, when he lists the symbols of logic, he always includes the symbols of the calculus of classes among them (and not only those symbols that are common to both calculi, such as '$\supset$'). See [Peano, 1889a, p. 28]/[1973, p. 103] and [Peano, 1900, p. 311].

[50] Note that, in contemporary terms, Peano's formulas could be seen as:

$$\forall a Raa, \forall ab((Rab \wedge Rbc) \rightarrow Rac)$$
$$\forall abc(Rcfab \leftrightarrow (Rca \wedge Rcb))$$
$$\forall abc(Rgabc \leftrightarrow (Rac \wedge Rbc)),$$

where 'R' is a binary relation symbol, and 'f' and 'g' are binary function symbols.

Despite the unprecedented character of Peano's account, he does not seem to make much of it. After all, the aforequoted passage belongs to a footnote in a review, and Peano might have thought that he is only reformulating Schröder's proof. More importantly, Peano does not have a systematic goal in mind; in this context, he shows no interest in the formalisation of the whole calculus of classes or in any metatheoretical consideration besides the independence of this specific distributive law.

In fact, Peano never fully exploited the formalisation of fundamental propositions in the calculus of classes. He restricted these kinds of considerations to independence arguments. Almost ten years after his reformulation of Schröder's proof of the independence of the distributive law, in 'Formules de logique mathématique', Peano considers again the independence of the distributive law which, on this occasion, is also an axiom of his calculus of classes [1900, p. 336]. In Peano's words:[51]

$$(\mathrm{P3}\cdot 01) \qquad a, b, c \,\varepsilon\, \mathrm{Cls}\,. \supset .\, a(b \cup c) \supset ab \cup ac \qquad \mathrm{Pp}$$

> This P [proposition] is not a consequence of the previous P [Propositions]. To recognise its independence, it suffices to give the signs Cls, $\cap$, $\cup$ an interpretation that satisfies the previous P, but not this one. Consider points; by Cls we indicate the convex classes of points, namely the u such that $\mathrm{Med}\, u = u$; the sign $\cap$ retains its value; then, by Df [Definition] $1 \cdot 0$, $a \cup b$ indicates "the smallest convex class that contains a and b". It is easy to see that the preceding propositions of §$\cup$ remain, and also the duals, but not the new ·01 [P3 · 01]. Therefore, following the organisation we have chosen here, we must consider it as a "primitive proposition". [Peano, 1900, p. 336]

Peano takes primitive and derived symbols of the calculus of classes, such as '$\cup$' or '$\cap$', and considers them as reinterpretable relation symbols. He also suggests an alternative domain of interpretation of the variables. Note that although the first occurrence of '$\supset$' in $(\mathrm{P3}\cdot 01)$ corresponds to a conditional and is thus a logical symbol, the second occurrence corresponds to the subsumption relation and is specific to the calculus of classes. Peano does not suggest a reinterpretation of '$\supset$' which retains its meaning as a relation between classes; in fact, had he given such an interpretation, he would have realised the inconvenience in this context of using a single symbol both as a logical symbol and as a relation symbol, i.e., to express both a logical relation and a relation between classes, as he had done in his review of the first volume of Schröder's *Vorlesungen über die Algebra der Logik* [1891e]. As a result, only some of the non-logical symbols of $(\mathrm{P3}\cdot 01)$ are reinterpreted and thus the formalisation of this axiom is only partial. Accordingly, Peano does not achieve a complete abstraction from the mathematical theory – i.e., the calculus of classes – to which this primitive proposition belongs. In the previous section, I explained that in *Principii di Geometria*, Peano uses

[51] The interpretation Peano puts forward to show the independence of $(\mathrm{P3}\cdot 01)$ is almost the same to one of the two he had given in [Peano, 1891e, p. 118]. However, on this occasion he does not perform the preliminary step of sharply separating the logical and the class-theoretical uses of the symbols '$\cup$' and '$\supset$' in the axiom.

the membership relation symbol 'ϵ' as a non-logical constant. However, in contrast to his methodology in the review to the first volume of Schröder's *Vorlesungen über die Algebra der Logik*, neither in *Principii di Geometria* nor in 'Formules de logique mathématique' does Peano provide specific symbols for the logical connectives; he represents them using '$\cap$', '$\cup$' and '$-$', which are also used to express class-theoretic operations.

Peano developed a logical language by means of which quantification can be adequately rendered. This logical language can be enlarged with symbols – including some of those belonging to the language of the calculus of classes – which are effectively used as non-logical constants. However, the use of symbols such as '$\supset$', '$\cap$' or '$\cup$' to express both logical relations and also class-theoretical relations, blurs the distinction between logical and non-logical symbols.[52] This exposes Peano's lack of a complete separation of interpreted and formal language in 'Formules de logique mathématique' [1900]. This could be explained by Peano's motivations. After all, his proofs of the independence of specific axioms of geometry or the calculus of classes do not require a complete separation between logical and class-theoretical language. According to a contemporary perspective, the calculus of classes is not a logical theory, but – as I have suggested in Footnote 49 – there is no reason to think that Peano also held this idea.

The technical tools required for metatheoretical investigations increased with the growing sophistication of the questions studied. The first example of the use of a formal language proper can be found in 1915. In 'Über Möglichkeiten im Logikkalkül' [1915] Löwenheim effectively formalises all the axioms of the calculus of classes in a first-order language, and such a formalisation is instrumental for the proof of the result known as the Löwenheim-Skolem theorem.[53] See, for instance, the first two axioms of the

[52]Peano does not confuse the two meanings attributed to the symbols of his mathematical logic. See, for instance, how he introduces the symbol of the intersection between classes in 'Formules de logique mathématique':

$$a, b \in \mathrm{Cls} \,.\supset: x \in a \cap b \,.=.\, x \in a \,.\cap.\, x \in b.$$

> The equality is a Df? (possible definition), because the sign $\cap$ occurs in the first member between Cls [classes], and in the second between P [propositions]. If its value between P is assumed to be known, the value of the formula $x \in ab$ will be deduced [...].
>
> Conversely, if the product ab of two Cls is considered a primitive idea, the value of the logical product between the P $x \in a$ and $x \in b$ will be deduced. However, the Hp [hypothesis] $a, b \in$ Cls, by P2 $\cdot$ 0 [$a \in \mathrm{Cls} \,.\supset: x, y \in a \,.=.\, x \in a \,.\, y \in a$] already is the logical product of two P. [Peano, 1900, pp. 324–325]

Peano wants to take advantage of the analogy between the calculus of classes and the sentential calculus, and encapsulate in a single symbol what is expressed informally as two different relations. See also [Peano, 1896–1897, p. 573]/[1973, p. 197].

[53]On Löwenheim's formalisation of the axioms of the calculus of classes, see [Badesa, 2004, pp. 60–71]. See also [Goldfarb, 1979, pp. 354–356]. Even though Löwenheim successfully formalises this calculus in [1915], Goldfarb claims that he "lacks a general notion – even of semantic kind – of a formalized mathematical theory" [1979, p. 355].

calculus of classes:

$$\text{(I)} \qquad a \mathrel{\underline{\in}} a,$$

$$\text{(II)} \qquad (a \mathrel{\underline{\in}} b)(b \mathrel{\underline{\in}} c) \mathrel{\underline{\in}} (a \mathrel{\underline{\in}} c),$$

and Löwenheim's formalisation [1915, p. 457]/[van Heijenoort, 1967, p. 240]:

$$\text{(I)} \qquad \prod_{a} s_{aa} = 1,$$

$$\text{(II)} \qquad \prod_{a,b,c} s_{ab} s_{bc} \mathrel{\underline{\in}} s_{ac},$$

where 's' is a binary relation symbol, '$\mathrel{\underline{\in}}$' the conditional and '$\prod_a$' is a universal quantifier that bounds a.

By formalising the calculus of classes, Löwenheim shows that the theory of relatives – developed by Schröder in the third volume of *Vorlesungen über die Algebra der Logik* [1895] – includes a formal language. The relation symbol 's' of this formal language, which takes the place of the relation symbol '$\mathrel{\underline{\in}}$' (when it is used as a relation symbol that belongs to the language of the calculus of classes), is clearly uninterpreted and can thus be considered a true non-logical constant.[54]

6. Concluding remarks

The profound changes that shaped the evolution of late nineteenth-century mathematics reflect the innovative work of mathematicians of that time. Peano's contributions put him at a turning point, in which a vindication of rigour imposed axiomatic presentations that, in turn, eased the investigation of metatheoretical issues. These innovations required in many cases the acquisition of a structuralist mindset. Moreover, they were not the result of the work of a single mathematician. Peano's work was informed not only by other Italian mathematicians, but also by German mathematicians such as Pasch, Grassmann, and Dedekind. In this paper, by studying Peano's construction of geometry and arithmetic, I have provided textual evidence that supports the claim that Peano was in several important respects a methodological structuralist.

In the first part of the paper, I have argued that in his symbolisations of arithmetic and geometry Peano focussed on the specification of the relational features of the systems of objects of these theories. For Peano, the primitive notions of geometry and arithmetic are specific entities, but cannot be defined or even elucidated. Accordingly, arithmetic and geometry have to begin in axioms, which express the structural properties of the system of objects constituted by the primitive notions. In this sense, it can be said that Peano fulfils the first requirement of a minimal methodological structuralism.

[54]According to what has been discussed – and particularly in the light of the comparison between Peano and Löwenheim's formalisations – I cannot agree with Cassari's claim that "Peano's [proposal] [...] involves – and this is actually the first time it has happened – two fundamental *metalogical* concepts: that of *formal language* and that of the *model* of a formal system" [2011, p. 145]. Unfortunately, Cassari does not provide textual evidence in support of his claim.

In the second part of the paper, I have claimed that Peano's construction of arithmetic, despite substantial similarities, should be distinguished from Dedekind's. First, for Peano, the system of natural numbers is given and not created by abstraction. More importantly, Peano's construction of arithmetic starts with axioms and not with explicit definitions and, in fact, his axiomatisation should not be identified with a higher-level definition such as Dedekind's definition of a simply infinite system. The terms involved in Peano's axiomatisation of arithmetic are not presented as bound variables but as non-logical constants. Therefore, his approach can be better framed as a schematic axiomatisation.

Lastly, by studying Peano's use of this schematic axiomatisation in the independence proofs, I have justified that Peano considered a multitude of systems that have the structural properties expressed by the axioms. He adopted a deductivist approach according to which the axioms are seen as formal postulates. Moreover, his proofs of independence involve considering alternative interpretations of the primitive terms which, in turn, determine systems that satisfy the axioms. Peano thus also satisfied, to a great extent, the second requirement of a minimal methodological structuralism. The symbolisations of arithmetic and geometry Peano provided are, in the contexts of these independence proofs, instrumental for the development of a model-theoretical point of view, and also show that Peano fell short of producing a formal language.

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