REVIEW OF POLLARD (2022) *ERNST SCHRÖDER ON ALGEBRA AND LOGIC*

JOAN BERTRAN-SAN-MILLÁN

It is widely accepted that Ernst Schröder's works on logic have been understudied. The fact that virtually none of Schröder's writings have been translated into English has certainly played a part in this neglect. Until recently, only Brady's *From Peirce to Skolem* (2000) had introduced Schröder to English readership, with translations of parts of the third volume of *Vorlesungen über die Algebra der Logik* (1895) (hereafter, *Vorlesungen*). In this respect, the publication of Pollard's volume, containing English translations of the most prominent of Schröder's early works, is a welcome and necessary contribution to the field.

After a brief Translator's Introduction, *Ernst Schröder on Algebra and Logic* provides translations of *Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende* (1873) (hereafter, *Lehrbuch*), *Der Operationskreis des Logikkalkuls* (1877a) (hereafter, *Operationskreis*), and 'Note über den Operationskreis des Logikcalculs' (1877b). The *Lehrbuch*, as its title indicates, was conceived as a mathematics textbook, but also engages in philosophical reflection. It presents number theory as the main constituent of pure mathematics. The *Operationskreis* inaugurates Schröder's focus on logic, and specifically on the algebra of logic: it is a short reformulation of Boolean logic. These works show clear and explicit influence from Grassmann's *Lehrbuch der Arithmetik für föhere Lehranstalten* (1861), and the 1877 treatise from Boole's *An Investigation of the Laws of Thought* (1854).[1] Their historical relevance is probably overshadowed, however, by the monumental *Vorlesungen*. Nevertheless, Schröder's *Lehrbuch* and *Operationskreis* also contain substantial contributions, as I will elaborate in what follows.

Mancosu (2016, 156, fn. 7) reflects on the similarities between Frege's and Schröder's characterisations of numbers. One of the fundamental elements of Frege's conception of number in *Grundlagen der Arithmetik* (1884) is the claim that a determination of number is relative to a concept. Moreover, a foundational question posed by Frege in his early works on logic is the generality of arithmetic and its relation to the generality of geometry; this was in fact a significant mathematical concern in the late nineteenth century and involves the relationship between complex analysis and geometry.[2] By 1873, as Mancosu points out, Schröder anticipates both elements (1873, 4–6).[3] Frege had already noted the similarities of his approach to Schröder's in *Grundlagen*, where he mentions Schröder's *Lehrbuch* in a footnote after introducing what is nowadays referred to as 'Hume's Principle' (1884, §63, 74).[4] Indeed, in the first sections of the *Lehrbuch*, by reflecting on the notions of

[1]Schröder was not alone among early modern mathematicians devoted to logic in being strongly influenced by Grassmann. Other significant examples are Peano and Frege.

[2]A geometrical foundation of analysis, and the development of a general notion of function lie at the core of this question, and were approached divergently by Weierstrass and Riemann. See (Tappenden, 1995, 2006).

[3]Page numbers in all quotes of (Schröder, 1873, 1877a,b) correspond to the reviewed volume.

[4]In Frege's words:

> Hume long ago mentioned such a means: "When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal." This opinion, that numerical equality or identity must

counting and equinumerosity, Schröder establishes criteria for the equality between the numbers (*Anzahlen*) assigned to the members of two collections (1873, 7–17): "If it is possible to connect one-to-one [...] each thing of one kind with a thing of another kind with no thing of either kind left out, then we say that the things of the one kind are equal in quantity [*in derselben Anzahl*] to those of the other kind." (1873, 7). Schröder explicitly limits this definition to finite collections (1873, 7, 17). For 1873, this limitation is not surprising, since Cantor's work on transfinite numbers was as yet unpublished.

Schröder's conception of equinumerosity changed in at least two significant respects from 1873 to the publication of the third volume of *Vorlesungen* (1895). First, as Mancosu (2016, 160, fn. 10) observes, Schröder assigns numbers to the members of a collection rather than to the collection itself in the *Lehrbuch* (1873, 7, 16). However, in his mature works Schröder explicitly assigns numbers to sets or systems (1895, 598–599) (see also (Schröder, 1899, 56–57)). Second, as a means to incorporate Dedekind's and specially Cantor's work on similarity between infinite systems and transfinite numbers, respectively, Schröder distinguishes between two situations in which the sizes of two sets can be compared (1895, 598–599). On the one hand, there is a general notion of equipollence (*Gleichmächtigkeit*), which applies to two sets that can be connected by a one-one correlation. This notion can be applied to infinite sets, but it is independent of the concept of number.[5] On the other hand, a more restricted notion, that of equinumerosity (*Gleichzahligkeit*), is relative to finite sets, and refers explicitly to two sets containing the same number of elements. In 'On Pasigraphy' (1899, 54–57), Schröder provides separate formal definitions of these two notions.

In the second chapter of the *Lehrbuch*, after discussing the basic properties of the sum and the product operations, Schröder introduces the law of distribution of the product over the sum (1873, 77):

$$a(b+c) = ab + ac. \tag{1}$$

Given the commutativity of the product, this law is equivalent to $(a+b)c = ac+bc$, which Schröder also presents as a distributive law. This is, of course, a well known arithmetical law, of which H. Grassmann had given a proof by induction in *Lehrbuch der Arithmetik für höhere Lehranstalten* (1861, §4, 21–22).

Although the *Lehrbuch* has several pages devoted to logic, Schröder was acquainted with R. Grassmann's *Formenlehre oder Mathematik* (1872) only in the later stages of composing the 1873 textbook (1873, 136, fn. 19). Moreover, he only became familiar with Boole's logical work after the publication of the *Lehrbuch* (see (Peckhaus, 2010, 228)). Boole includes the logical counterpart of (1) as a law in *An Investigation of the Laws of Thought* (1854, 33), as does Jevons in like fashion in *Pure logic* (1864, 26). In 'On an Improvement in Boole's Calculus of Logic' (1867), Peirce offers (1) as well as the law of distribution of the logical addition over the logical multiplication:[6]

$$ab + c = (a+c)(b+c), \tag{2}$$

in the logical calculus, and then provides the first logical proofs of each, which amounts to reasoning based on the fact that the terms equated in both laws refer to the same regions of the universe (1867, 251–252). In *Operationskreis*, Schröder also

> be defined in terms of one-one correlation, seems in recent years to have gained widespread acceptance among mathematicians (Frege, 1884, §63, pp. 73–74)

[5]This seems to contribute to Mancosu's claim (2016, 163–164) that for Schröder—in the *Lehrbuch*—infinite collections do not have a number.

[6]Formula (2) is interpreted only as a logical law, since—as Schröder is well aware (1880, 85) and (1890, §12, 285–286)—it is invalid in arithmetic.

mentions both distributive laws: (1) as an axiom and (2) as a theorem of the logical calculus (1877a, 300). He presents diagrams that display the regions of the universe corresponding to both terms of each equation (1877a, 301), but he does not show how (2) could be obtained from (1). Despite quoting Boole and Jevons in *Operationskreis*, it is very unlikely that Schröder had read any of Peirce's works before composing the 1877 treatise (see (Hailperin, 2004, 371–372)); in fact, it seems that he came up with (2) independently, motivated by the dualistic presentation of *Operationskreis*. This is confirmed in 'Note über den Operationskreis des Logikcalculs', where the distributive laws are furnished as "one noteworthy example" of the "*perfect* dualism" between addition (and subtraction) and multiplication (and division) (1877b, 335–336).[7]

In 'On the Algebra of Logic' (1880), Peirce lays down the basis of an axiomatisation of the calculus of classes, which Schröder completes in the first volume of *Vorlesungen* (1890). In 1880, Peirce develops the calculus of classes on the basis of the inclusion relation—which he identifies with the copula and the consequence relation, and he claims that the distributive laws are theorems of such calculus (1880, 33). After reading Peirce's (1880), Schröder reflected on the following laws:

$$a(b+c) \Subset ab + ac \tag{3}$$

$$(a+b)(a+c) \Subset a + bc \tag{4}$$

In a short conference paper, Schröder claims that (4)—although he should have said (3)—is independent of the calculus of classes (1884, 412). And then, in the first volume of *Vorlesungen*, Schröder demonstrates that (3) is independent of the first seven axioms of the calculus of classes (1890, §12, 282–298, App. 4 & 6: 617–632, 647–699). The calculus thus requires the addition of an axiom similar to (3), from which both (3) and (4) could be derived. (1890, §12, 293). The proof and justification of independence of the distributive laws stimulated a dispute between Peirce and Schröder that ultimately relied on the basic constituents of the calculus of classes.[8]

Pollard's book has now made these contributions accessible to English scholarship. He has produced a clear translation of Schröder's writings, which is not an easy task given the latter's convoluted style. Surprisingly, Pollard neither includes translations of the Foreword nor the Appendix of the *Lehrbuch*.[9] The Translator's Introduction is short and technically oriented, but informative. That said, more context about Schröder's biography and his work—besides the *Lehrbuch* and *Operationskreis*—would have been welcome. The Translator's Introduction does not even include a list of Schröder's writings.

Pollard's goal of offering a translation that recovers the sense of Schröder's text but avoids the awkwardness of a literal rendering is understandable, but I would have appreciated an explanation of the most significant choices made in this translation. As a final note, sometimes the editorial apparatus can become

[7]Schröder is quite explicit about his aim to take advantage of the duality between logical addition and multiplication:

> If, in a logically valid general formula, we everywhere interchange plus-signs and multiplication-signs, minus-signs and division-signs, and replace the symbols 0 and 1 with one another, the result will once again be a correct formula. (Schröder, 1877a, 294)

As Pollard suggests, this account had been already hinted at in the *Lehrbuch* (1873, 136, fn. 19), where Schröder refers to R. Grassmann's *Die Formenlehre oder Mathematik* (1872).

[8]On the dispute between Peirce and Schröder regarding the distributive laws, see (Badesa, 2004, 21–25). See also (Houser, 1991).

[9]Although Pollard acknowledges this fact in the Translator's Introduction, no explanation is given for this omission. Nevertheless, Pollard quotes passages from Schröder's Foreword in the Translator's Introduction.

distracting. The abundance of footnotes to Schröder's text often documents the translator's interpretative doubts, but it is not entirely clear to me how helpful they might be.

Leaving these shortcomings aside, *Ernst Schröder on Algebra and Logic* should immediately become a key reference for everyone interested in the development of modern logic. Schröder's early works are a good example of the close connection between mathematical problems and logical resources that characterized the final quarter of the nineteenth century. The careful study of these writings will not only enrich the field at large, but more specifically the scholarship on Frege and Peano.

References

Badesa, C. (2004). *The Birth of Model Theory.* Princeton: Princeton University Press.

Boole, G. (1854). *An Investigation of the Laws of Thought.* Cambridge: Macmillan.

Brady, G. (2000). *From Peirce to Skolem. A neglected chapter in the history of logic.* Studies in the History and Philosophy of Mathematics. Amsterdam: Elsevier.

Frege, G. (1884). *Die Grundlagen der Arithmetik: Eine logisch-matematische Untersuchung über den Begriff der Zahl.* Breslau: Wilhelm Koebner.

Grassmann, H. (1861). *Lehrbuch der Arithmetik für höhere Lehranstalten.* Berlin: Adolph Enslin.

Grassmann, R. (1872). *Die Begriffslehre oder Logik. Zweites Buch der Formenlehre oder Mathematik.* Stettin: R. Grassmann.

Hailperin, T. (2004). Algebraic Logic 1685–1900. In Gabbay, D. M.; Woods, J. (Eds.), *Handbook of the History of Logic*, vol. 3: The Rise of Modern Logic: From Leibniz to Frege, Amsterdam: Elsevier North Holland, pp. 323–388.

Houser, N. (1991). Peirce and the Law of Distribution. In Drucker, T. (Ed.), *Perspectives on the History of Mathematical Logic*, Basel: Birkhäuser, pp. 10–32.

Jevons, W. S. (1864). *Pure logic or the logic of quality apart from quantity.* London: Stanford.

Mancosu, P. (2016). *Abstraction and Infinity.* Oxford: Oxford University Press.

Peckhaus, V. (2010). Robert and Hermann Grassmann's influence on the history of formal logic. In Petsche, H. J.; Lewis, A.; Liesen, J.; Russ, S. (Eds.), *From Past to Future: Graßmann's Work in Context*, Basel: Springer, pp. 221–228.

Peirce, C. S. (1867). An Improvement in Boole's Calculus of Logic. *Proceedings of the American Academy of Arts and Sciences*, *7*, pp. 249–261.

Peirce, C. S. (1880). On the Algebra of Logic. *American Journal of Mathematics*, *3*, pp. 15–57.

Schröder, E. (1873). *Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende.* Leipzig: G. Teubner.

Schröder, E. (1877a). *Der Operationskreis des Logikkalkuls.* Leipzig: G. Teubner.

Schröder, E. (1877b). Note über den Operationskreis des Logikcalculs. *Mathematische Annalen*, *12*, pp. 481–484.

Schröder, E. (1880). Rezension: G. Frege, *Begriffsschrift. Zeitschrift für Mathematik und Physik*, *25*, pp. 81–94.

Schröder, E. (1884). Exposition of a Logical Principle, as Disclosed by the Algebra of Logic but Overlooked by the Ancient Logicians. In *Report ot the Fifty-third Meeting of the British Association for the Advancement of Science held at Southport in September 1883*, London: John Murray, p. 412.

Schröder, E. (1890). *Vorlesungen über die Algebra der Logik (exakte Logic)*, vol. 1. Leipzig: G. Teubner.

Schröder, E. (1895). *Vorlesungen über die Algebra der Logik (exakte Logic)*, vol. 3. Leipzig: G. Teubner.

Schröder, E. (1899). On Pasigraphy. Its Present State and the Pasigraphic Movement in Italy. *The Monist*, *9*, pp. 246–262.

Tappenden, J. (1995). Geometry and Generality in Frege's Philosophy of Arithmetic. *Synthese*, *102*, pp. 319–361.

Tappenden, J. (2006). The Riemannian Background to Frege's Philosophy. In Ferreirós, J.; Gray, J. J. (Eds.), *The Architecture of Modern Mathematics. Essays in History and Philosophy*, Oxford: Oxford University Press, pp. 97–132.